An embedding of ChuCors in $L$-ChuCors

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Abstract

An $L$-fuzzy generalization of the so-called Chu correspondences between formal contexts forms a category called $L$-ChuCors. In this work we show that this category naturally embeds ChuCors.

Key words: Formal Concept Analysis, Category theory, $L$-fuzzy logic

1 Preliminaries

Formal concept analysis (FCA) introduced by Ganter and Wille [6] has become an extremely useful theoretical and practical tool for formally describing structural and hierarchical properties of data with “object-attribute” character. Bělohlávek in [1, 2] provided an $L$-fuzzy extension of the main notions of FCA, such as context and concept, by extending its underlying interpretation on classical logic to the more general framework of $L$-fuzzy logic [7].

In this work, we aim at formally describing some structural properties of inter-contextual relationships [5, 11] of $L$-fuzzy formal contexts by using category theory [3], following the results in [12, 13]. The category $L$-ChuCors is formed by considering the class of $L$-fuzzy formal contexts as objects and the $L$-fuzzy Chu correspondences as arrows between objects.

The main result here is that $L$-ChuCors embeds the category ChuCors. This result is illustrated by showing different categories $L$-ChuCors built on different underlying truth-values sets $L$.

In order to make this contribution as self-contained as possible, we proceed now with the preliminary definitions of complete residuated lattice, $L$-fuzzy context, $L$-fuzzy concept and $L$-Chu correspondence.

Definition 1 An algebra $\langle L, \land, \lor, \otimes, \rightarrow, 0, 1 \rangle$ is said to be a complete residuated lattice if
Definition 2 Let \( L \) be a complete residuated lattice, an \( L \)-fuzzy context is a triple \( \langle B, A, r \rangle \) consisting of a set of objects \( B \), a set of attributes \( A \) and an \( L \)-fuzzy binary relation \( r \), i.e. a mapping \( r: B \times A \to L \), which can be alternatively understood as an \( L \)-fuzzy subset of \( B \times A \).

We now introduce the \( L \)-fuzzy extension provided by Bělohlávek [1], where we will use the notation \( Y^X \) to refer to the set of mappings from \( X \) to \( Y \).

Definition 3 Consider an \( L \)-fuzzy context \( \langle B, A, r \rangle \). A pair of mappings \( \uparrow: L^B \to L^A \) and \( \downarrow: L^A \to L^B \) can be defined for every \( f \in L^B \) and \( g \in L^A \) as follows:

\[
\uparrow f(a) = \bigwedge_{o \in B} (f(o) \to r(o, a)) \quad \downarrow g(o) = \bigwedge_{a \in A} (g(a) \to r(o, a))
\]

Lemma 1 Let \( L \) be a complete residuated lattice, let \( r \in L^{B \times A} \) be an \( L \)-fuzzy relation between \( B \) and \( A \). Then the pair of operators \( \uparrow \) and \( \downarrow \) form a Galois connection between \( \langle L^B; \subseteq \rangle \) and \( \langle L^A; \subseteq \rangle \), that is, \( \uparrow: L^B \to L^A \) and \( \downarrow: L^A \to L^B \) are anti-adjoint and, furthermore, for all \( f \in L^B \) and \( g \in L^A \) we have \( f \subseteq \downarrow \uparrow f \) and \( g \subseteq \uparrow \downarrow g \).

Definition 4 Consider an \( L \)-fuzzy context \( C = \langle B, A, r \rangle \). An \( L \)-fuzzy set of objects \( f \in L^B \) (resp. an \( L \)-fuzzy set of attributes \( g \in L^A \)) is said to be closed in \( C \) iff \( f = \uparrow \uparrow f \) (resp. \( g = \downarrow \downarrow g \)).

Lemma 2 Under the conditions of Lemma 1, the following equalities hold for arbitrary \( f \in L^B \) and \( g \in L^A \), \( \uparrow f = \uparrow \uparrow \uparrow f \) and \( \downarrow g = \downarrow \downarrow \downarrow g \), that is, both \( \uparrow \uparrow f \) and \( \downarrow \downarrow g \) are closed in \( C \).

Definition 5 An \( L \)-fuzzy concept is a pair \( \langle f, g \rangle \) such that \( \uparrow f = g \), \( \downarrow g = f \). The first component \( f \) is said to be the extent of the concept, whereas the second component \( g \) is the intent of the concept.

The set of all \( L \)-fuzzy concepts associated to a fuzzy context \( \langle B, A, r \rangle \) will be denoted as \( L\text{-FCL}(B, A, r) \).

An ordering between \( L \)-fuzzy concepts is defined as follows: \( \langle f_1, g_1 \rangle \leq \langle f_2, g_2 \rangle \) if and only if \( f_1 \subseteq f_2 \) if and only if \( g_1 \geq g_2 \).

Proposition 1 The poset \( (L\text{-FCL}(B, A, r), \leq) \) is a complete lattice where

\[
\bigwedge_{j \in J} \langle f_j, g_j \rangle = \langle \bigwedge_{j \in J} f_j, \uparrow \bigwedge_{j \in J} (\bigwedge_{j \in J} f_j) \rangle
\]

\[
\bigvee_{j \in J} \langle f_j, g_j \rangle = \langle \downarrow (\bigwedge_{j \in J} g_j), \bigwedge_{j \in J} g_j \rangle
\]
Finally, we proceed with the definition of $L$-Chu correspondences, for which we need the notion of $L$-multifunction.

**Definition 6** An $L$-**multifunction** from $X$ to $Y$ is a mapping $\varphi : X \to L^Y$. The set $L\text{-Mfn}(X,Y)$ of all the $L$-multifunctions from $X$ to $Y$ can be endowed with a poset structure by defining the ordering $\varphi_1 \leq \varphi_2$ as $\varphi_1(x)(y) \leq \varphi_2(x)(y)$ for all $x \in X$ and $y \in Y$.

**Definition 7** Consider two $L$-fuzzy contexts $C_i = (B_i, A_i, r_i), (i = 1, 2)$, then the pair $\varphi = (\varphi_l, \varphi_r)$ is called a correspondence from $C_1$ to $C_2$ if $\varphi_l$ and $\varphi_r$ are $L$-multifunctions, respectively, from $B_1$ to $B_2$ and from $A_2$ to $A_1$ (that is, $\varphi_l : B_1 \to L^{B_2}$ and $\varphi_r : A_2 \to L^{A_1}$).

The $L$-correspondence $\varphi$ is said to be a weak $L$-**Chu correspondence** if the equality $\hat{r}_1(\chi_{o_1}, \varphi_r(a_2)) = \hat{r}_2(\varphi_l(o_1), \chi_{a_2})$ holds for all $o_1 \in B_1$ and $a_2 \in A_2$. By unfolding the definition of $\hat{r}_i$ this means that

$$\bigwedge_{a_1 \in A_1} (\varphi_r(a_2)(a_1) \to r_1(o_1, a_1)) = \bigwedge_{o_2 \in B_2} (\varphi_l(o_2)(a_2) \to r_2(o_2, a_2)) \quad (2)$$

A weak Chu correspondence $\varphi$ is an $L$-**Chu correspondence** if $\varphi_l(o_1)$ is closed in $C_2$ and $\varphi_r(a_2)$ is closed in $C_1$ for all $o_1 \in B_1$ and $a_2 \in A_2$. We will denote the set of all $Chu$ correspondences from $C_1$ to $C_2$ by $L\text{-ChuCors}(C_1, C_2)$.

In the following definition and lemma, we introduce some connections between the right and the left sides of $L$-Chu correspondences.

**Definition 8** Given a mapping $\varpi : X \to L^Y$ we consider the following associated mappings $\varpi_* : L^X \to L^Y$ and $\varpi^* : L^Y \to L^X$, defined for all $f \in L^X$ and $g \in L^Y$ by

1. $\varpi_*(f)(y) = \bigvee_{x \in X} (f(x) \otimes \varpi(x)(y))$
2. $\varpi^*(g)(x) = \bigwedge_{y \in Y} \varpi(x)(y) \to g(y)$

**Lemma 3** Let $C_i = (B_i, A_i, r_i)$ for $i = 1, 2$ be $L$-fuzzy contexts. Let $\varphi = (\varphi_l, \varphi_r) \in L\text{-ChuCors}(C_1, C_2)$. Then

- for all $f \in L^{B_1}$ and $g \in L^{A_2}$, the following equalities hold
  $$\uparrow_2 (\varphi_* (f)) = \varphi^* (\uparrow_1 (f)) \quad \text{and} \quad \downarrow_1 (\varphi_{r*} (g)) = \varphi^*_r (\downarrow_2 (g))$$

- for all $o_1 \in B_1$ and $a_2 \in A_2$, the following equalities hold
  $$\varphi_l(o_1) = \uparrow_2 (\varphi^*_r (\downarrow_1 (\chi_{o_1}))) \quad \text{and} \quad \varphi_r(a_2) = \downarrow_1 (\varphi^*_l (\uparrow_2 (\chi_{a_2})))$$
The category $L$-ChuCors

We introduce now the category of $L$-Chu correspondences between $L$-fuzzy formal contexts as follows:

- **objects** $L$-fuzzy formal contexts
- **arrows** $L$-Chu correspondences
- **composition** $\varphi_2 \circ \varphi_1 : C_1 \to C_3$ of arrows $\varphi_1 : C_1 \to C_2$, $\varphi_2 : C_2 \to C_3$ ($C_i = \langle B_i, A_i, r_i \rangle$, $i \in \{1, 2\}$)
  - $(\varphi_2 \circ \varphi_1)_l : B_1 \to L^{B_3}$ and $(\varphi_2 \circ \varphi_1)_r : A_3 \to L^{A_1}$
  - $(\varphi_2 \circ \varphi_1)(o_1) = \|3\|_3 (\varphi_2 \star (\varphi_1)(o_1)))$
    
    $$\varphi_2 \star (\varphi_1)(o_1))(o_3) = \bigvee_{o_2 \in B_2} \varphi_2(o_2) \otimes \varphi_1(o_2)(o_3)$$
  - $(\varphi_2 \circ \varphi_1)_r(a_3) = \|1\|_1 (\varphi_1 \star (\varphi_2)(a_3)))$
    
    $$\varphi_1 \star (\varphi_2)(a_3))(a_1) = \bigvee_{a_2 \in A_2} \varphi_1(a_2) \otimes \varphi_2(a_2)(a_1)$$

**Theorem 1** $L$-fuzzy Chu correspondences between $L$-fuzzy formal contexts form a category with the composition defined above.

**Proof:** We just have to check the existence of identity arrows and the associativity of composition. The latter is just a matter of straightforward calculation, the identity arrows $\iota : C \to C$ are defined as follows for any given $L$-fuzzy context $C = \langle B, A, r \rangle$:

- $\iota_l(o) = \|\uparrow \| (\chi_o)$, for all $o \in B$
- $\iota_r(a) = \|\downarrow \| (\chi_a)$, for all $a \in A$.

3 $L$-ChuCors embeds ChuCors

In the following paragraph, we sketchily argue that ChuCors can be embedded in any of the extensions $L$-ChuCors where $L$ is a complete residuated lattice.

Assume that $(L_1, \land, \lor, \otimes, \rightarrow, 0, 1)$ and $(L_2, \land, \lor, \otimes, \rightarrow, 0, 1)$ are two complete residuated lattices, such that $L_2$ is a sublattice of $L_1$. Any $L_2$-fuzzy formal context $(B, A, r)$ satisfies that $r \in L_2^{B \times A} \subseteq L_1^{B \times A}$. This inclusion implies that the class of all objects of $L_2$-ChuCors is a subclass of $L_1$-ChuCors. Moreover, every concept constructed in $(B, A, r)$ by using the underlying logic provided by $L_2$ can be seen as well as a concept under the logic of $L_1$. As a result, the concept lattice $L_2$-$FCL(B, A, r)$ is a sublattice of the concept lattice $L_1$-$FCL(B, A, r)$.

The following example illustrates the previous results on the light of two particular cases for $L_1$.

**Example 1** Consider $L_1$ and $L_2$ the lattices shown to the left of the picture below, together with the two $L_2$-fuzzy formal contexts shown in the right.
Consider two complete residuated lattice(s) to be consisting of the infimum on $L_i$, together with its residual implication defined as $k \rightarrow l = \bigvee \{m \in L_i \mid m \land k \leq l\}$, for all $k,l,m \in L_i$ where $i \in \{1,2\}$. The concept lattices on the underlying logic of $L_1$ are shown in the pictures below, where the concepts in bold line are those in the frame associated to $L_2$.

The common $L_2$ and $L_1$-Chu correspondences are shown below:

$\varphi_l^{1,c,c} \varphi_r^{a,c,1} \varphi_l^{1,c} \varphi_r^{a,c} \varphi_l^{1,c} \varphi_r^{a,c}$

The following result formally states the general relation between $L_i$-ChuCors.

**Lemma 4** Let $C_1, C_2$ be the $L_2$-contexts. $L_2$-ChuCors($C_1, C_2$) $\subseteq$ $L_1$-ChuCors($C_1, C_2$).

It is easy to see that the connection of two $L_2$-Chu correspondences make a new $L_2$-Chu correspondence. In addition, the set of $L_2$-Chu correspondences between two $L_2$-contexts is a subset of all $L_1$-Chu correspondences between the same contexts. $L_2$-Chu correspondences form a category, so the set of arrows is closed under the connections of arrows, as a result the set of $L_1$-Chu correspondences is closed under connections of $L_2$-Chu correspondences. Thus, we have just proved the following.

**Lemma 5** Let $C_i$ for $i \in \{1,2,3\}$ be two $L_2$-contexts. For every $L_2$-Chu correspondence $\varphi \in L_2$-ChuCors($C_1, C_2$) and $\psi \in L_2$-ChuCors($C_2, C_3$) holds $\psi \circ \varphi \in L_1$-ChuCors($C_1, C_3$).

In consequence, we can state
Theorem 2 Under the environment hypotheses of this section, the category $L_2$-ChuCors naturally embeds in $L_1$-ChuCors.

As the category ChuCors of classical Chu correspondences are defined on classical, two-valued logic, which is a special case of any logic defined on complete residuated lattice, we obtain

Corollary 1 The category ChuCors naturally embeds in $L$-ChuCors

References


