Linking $L$-Chu correspondences and completely lattice $L$-ordered sets

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Abstract Continuing our categorical study of $L$-fuzzy extensions of formal concept analysis, we provide a representation theorem for the category of $L$-Chu correspondences between $L$-formal contexts and prove that it is equivalent to the category of completely lattice $L$-ordered sets.

Keywords Concept lattice, Category theory, Adjunction, Galois connection, Fuzzy logic, Equivalence functor

1 Introduction

Category theory has become important in many areas of modern mathematics (either as a research area per se or as a tool for doing mathematics) and computer science (as a means to unifying several approaches of abstract machines, or type theories, etc), although its use in other areas of computer science tend to find resistance, due to the reluctance to admit high levels of abstraction; on the other hand, Formal Concept Analysis (FCA) has become an extremely useful theoretical and practical tool for formally describing structural and hierarchical properties of data with “object-attribute” character, and this applicability justifies the need of a deeper knowledge of its underlying mechanisms: and one important way to obtain this extra knowledge turns out to be via generalization and abstraction.

Goguen argues in [22] that research on concepts should be thoroughly interdisciplinary, and in particular, should transcend the boundaries between sciences and humanities. One of the tools that he proposes is precisely category theory as a unifying language capable of merging different apparently disparate approaches. Not trying to reach such an ambitious goal (at least on the short/mid term), this paper continues...
previous work of the authors of a categorical study of FCA, and deals with an extremely
general form of $L$-fuzzy FCA, based on categorical constructs and $L$-fuzzy sets.

The introduction of generalization and abstraction in FCA, as in many other re-
search areas, may lead to new theoretical and applied results. For instance, concerning
the use of fuzzy (or, in some cases, $L$-fuzzy) FCA, one can see papers ranging from reso-
lution of fuzzy relational equations [2] and ontology merging [12], to applications to the
Semantic Web by using the notion of concept similarity or rough sets [17,18], and from
noise control in document classification [32] to the development of recommender sys-
tems [15], or the study of fuzzy databases, in areas such as functional dependencies [37]
or data mining in terms of closure systems [10].

Theoretically, several approaches have been presented for generalizing the frame-
work and the scope of FCA and, nowadays, one can see works which extend it by
using ideas from fuzzy set theory [1,3], rough set theory [30,31,42], the multi-adjoint
framework [33,34,36], or possibility theory [16], or heterogeneous approaches in which
concept lattices are based on Galois connections allowing to analyse object-attribute
models with different structures for truth values of attributes [11,35].

The use of category theory to study of ideas related to FCA has proliferated in the
recent years; for instance, the Information Flow Framework [25] provides a framework
for ontology development making it possible to communicate between categorical and
FCA formalisms, or the study of concept structures done by Hitzler et al [23, 24]
applying categorical methods to define the notion of approximable structure (borrowed
from the field of denotational semantics), or the categorical study of fuzzy Galois
connections [20] allowing for presenting its theory in a more succinct way, and providing
a useful method to study the links between the commutative and the non-commutative
worlds, or a more abstract study of the concept lattice functors [40] including the
relationship between contexts, closure spaces, and complete lattices, or the categorical
view of generalized concept lattices [26].

The categorical treatment of morphisms as fundamental structural properties has
been advocated by [29] as a means for the modelling of data translation, communica-
tion, and distributed computing, among other applications. Our approach broadly
focuses on the research line which links the theory of Chu spaces with concept lat-
tices [44,45]; in the latter, it is shown that the notion of state in Scott’s information
system corresponds precisely to that of formal concepts in FCA with respect to all fi-
nite Chu spaces, and the entailment relation corresponds to association rules (another
link between FCA with database theory) and, specifically, on the identification of the
categories associated to certain constructions.

Our approach is particularly based on the notion of Chu correspondences between
formal contexts, developed by Mori, which we briefly sketch below:

In [38], the author focused on the great number of structures having certain duality
which can be formalized in terms of a formal context $(B,A,r)$. Homomorphisms
between two of these structures, with contexts $(B_i,A_i,r_i)$ for $i \in \{1,2\}$, induce Chu mapp-
ings, that is pairs of mappings $\varphi: B_1 \rightarrow B_2$ and $\psi: A_2 \rightarrow A_1$ such that $r_2(\varphi(b_1),a_2) =
\varphi(r_1(b_1,\psi(a_2)))$; note the similarity with the adjoint property of isotone Galois con-
nexions. A functor, the Galois functor, can be naturally defined from the category of Chu mapp-
ings and the category of join-preserving mappings between complete lattices; unfortunately, this functor is neither full (surjective) nor faithful (injective). The main
contribution of [38] was the introduction of the notion of Chu correspondence and
proving, on the one hand, the fullness and faithfulness of the Galois functor and, on
the other hand, the $*$-autonomous structure of the category of Chu correspondences.
Previous work in this categorical approach has been already developed by the authors. In [28], the notion of \(L\)-Chu correspondence between \(L\)-contexts was introduced; in addition, the resulting set of \(L\)-Chu correspondences was shown to be a complete lattice anti-isomorphic to that of \(L\)-bonds between formal contexts. More recently [27], the authors started the categorical study of \(L\)-contexts and their morphisms by introducing the category \(L\)-ChuCors, having \(L\)-contexts as objects and \(L\)-Chu correspondences as morphisms, providing a further abstraction with the aim of formally describing structural properties of intercontextual relationships. In addition, in that paper it was proved that the resulting category is \(*\)-autonomous and, therefore, its underlying logic is classical linear logic [5, 39].

In the present work, we continue our study of \(L\)-ChuCors, seen as a common categorical umbrella for several fuzzy extensions of the classical notion concept lattice, initiated mainly by Bělohlávek [6–9], who extended the underlying interpretation on classical logic to the more general framework of \(L\)-fuzzy logic [21].

Pictorially, we can represent the contribution of this work as the right arrow in Figure 1, which somehow closes the initial study of \(L\)-ChuCors, in that we already have completed the picture of the behavior of Chu correspondences in an \(L\)-fuzzy environment. Specifically, the main result in this work is a constructive proof of the equivalence between the category \(L\)-ChuCors and a category of completely lattice \(L\)-ordered sets (\(L\)-CLOS\(^1\)) with isotone Galois connections between them. This result, on the one hand, reinforces the notion of \(L\)-CLOS as the most adequate fuzzy version of complete lattice, since in the crisp case the equivalence is with the category of join-preserving maps between complete lattices; on the other hand, paves the way for future work on finding further connections following the thread of \(L\)-CLOS; an interesting possibility might be studying the topic of approximable concepts, because of the existing relationship between them and \(L\)-CLOS [13].

In order to obtain a reasonably self-contained document, Section 2 introduces the basic definitions concerning the \(L\)-fuzzy extension of formal concept analysis, as well as those concerning \(L\)-Chu correspondences; then, the categories associated to \(L\)-formal contexts and \(L\)-CLOS are defined in Section 3 and, finally, the proof of equivalence is in Section 4.

\(^1\) Although the proper acronym should be CLLOS, we prefer to use the prefix \(L\) in the acronym to better reflect that we are working on an \(L\)-fuzzy extension.
2 Preliminaries

2.1 Basics of \(L\)-fuzzy FCA

In this section we introduce the preliminary definitions of \(L\)-fuzzy FCA and the theory of completely \(L\)-lattice ordered sets. In this respect, we are assuming the same motivations used in [8,9].

Definition 1 An algebra \(\langle L, \land, \lor, \otimes, \to, 0, 1 \rangle\) is said to be a complete residuated lattice if

- \(\langle L, \land, \lor, 0, 1 \rangle\) is a complete lattice with the least element 0 and the greatest element 1,
- \(\langle L, \otimes, 1 \rangle\) is a commutative monoid,
- \(\otimes\) and \(\to\) are adjoint, i.e. \(a \otimes b \leq c\) if and only if \(a \leq b \to c\), for all \(a, b, c \in L\), where \(\leq\) is the ordering in the lattice generated from \(\land\) and \(\lor\).

For a good overview of the theory of complete residuated lattices, the reader is referred to [19].

Definition 2 Let \(L\) be a complete residuated lattice, an \(L\)-fuzzy context is a triple \(\langle B, A, r \rangle\) consisting of a set of objects \(B\), a set of attributes \(A\) and an \(L\)-fuzzy binary relation \(r\), i.e. a mapping \(r: B \times A \to L\), which can be alternatively understood as an \(L\)-fuzzy subset of \(B \times A\). The set of all \(L\)-sets of objects from \(B\) will be denoted by \(L^B\), and similarly for any base set.

Definition 3 Consider an \(L\)-fuzzy context \(\langle B, A, r \rangle\). Mappings \(\uparrow: L^B \to L^A\) and \(\downarrow: L^A \to L^B\) can be defined for every \(f \in L^B\) and \(g \in L^A\) as follows:

\[
\uparrow (f)(a) = \bigwedge_{o \in B} \{ f(o) \to r(o, a) \} \quad \downarrow (g)(o) = \bigvee_{a \in A} \{ g(a) \to r(o, a) \} \tag{1}
\]

Definition 4 An \(L\)-fuzzy concept is a pair \(\langle f, g \rangle\) such that \(\uparrow (f) = g\) and \(\downarrow (g) = f\). The first component \(f\) is said to be the extent of the concept, whereas the second component \(g\) is the intent of the concept.

The set of all \(L\)-fuzzy concepts associated to a fuzzy context \(\langle B, A, r \rangle\) will be denoted as \(L\text{-FCL}(B, A, r)\).

An ordering between \(L\)-fuzzy concepts is defined as follows: \(\langle f_1, g_1 \rangle \leq \langle f_2, g_2 \rangle\) if and only if \(f_1 \subseteq f_2\) (namely, \(f_1(o) \leq f_2(o)\) for all \(o \in B\)) if and only if \(g_1 \supseteq g_2\) (that is, \(g_1(a) \geq g_2(a)\) for all \(a \in A\)).

Example 1 Consider two \(L\)-contexts \(C_1\) and \(C_2\), where \(L = \{0, 0.5, 1\}\). Any value from \(L\) in any cell of the following tables represents a relationship of the corresponding object-attribute pair. It is a formalization of the information about the degree that an object has some attribute or, conversely, how any attribute is shared by some object.

\[
\begin{array}{cccc}
C_1 & a_{11} & a_{12} & a_{13} & a_{14} \\
\hline
a_{11} & 1 & 1 & 0.5 & 0 \\
a_{12} & 1 & 0.5 & 1 & 0.5 \\
\end{array}
\quad
\begin{array}{cccc}
C_2 & a_{21} & a_{22} & a_{23} \\
\hline
a_{21} & 1 & 1 & 0 \\
a_{22} & 0.5 & 1 & 1 \\
a_{23} & 0 & 0.5 & 1 \\
\end{array}
\]
If we consider Łukasiewicz logic connectives \( \langle \otimes, \rightarrow \rangle \), defined as
\[
\begin{align*}
k \otimes m &= \max\{0, k + m - 1\} & k \rightarrow m &= \min\{1, 1 - k + m\}
\end{align*}
\]
and with derivation operators \( \langle \uparrow, \downarrow \rangle \) defined in (1) above we obtain the set of all \( L \)-concepts of \( C_1 \) and \( C_2 \) that are shown in the following tables. Each row represents one \( L \)-concept. For the sake of readability, we will denote \( L \)-concepts of \( C_1 \) as \( p_1, \ldots, p_5 \) and \( L \)-concepts of \( C_2 \) as \( q_1, q_2, q_3 \).

<table>
<thead>
<tr>
<th>( L )-FCL(( C_1 ))</th>
<th>( o_{11} )</th>
<th>( o_{12} )</th>
<th>( a_{11} )</th>
<th>( a_{12} )</th>
<th>( a_{13} )</th>
<th>( a_{14} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>( p_2 )</td>
<td>1</td>
<td>0.5</td>
<td>1</td>
<td>1</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>( p_3 )</td>
<td>0.5</td>
<td>1</td>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>( p_4 )</td>
<td>0.5</td>
<td>0.5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>( p_5 )</td>
<td>0</td>
<td>0.5</td>
<td>1</td>
<td>1</td>
<td>1</td>
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</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( L )-FCL(( C_2 ))</th>
<th>( o_{21} )</th>
<th>( o_{22} )</th>
<th>( o_{23} )</th>
<th>( a_{21} )</th>
<th>( a_{22} )</th>
<th>( a_{23} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>( q_2 )</td>
<td>1</td>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
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<tr>
<td>( q_3 )</td>
<td>1</td>
<td>0.5</td>
<td>0</td>
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</tr>
</tbody>
</table>

There are four extremal \( L \)-concepts, namely, \( p_1, p_5, q_1 \) and \( q_3 \) formalizing the situation in which the (crisp) set of all objects has the whole set corresponding \( L \)-set of common attributes or, vice versa, that all attributes are shared by the corresponding \( L \)-set of objects. There are also examples of \( L \)-concepts covering the information about some \( L \)-set of objects satisfying an \( L \)-set of common attributes, for instance \( p_2 \). Finally, concept \( p_4 \) has objects \( o_{11} \) and \( o_{12} \) both with membership degree 0.5 which share the common attributes \( a_{11}, a_{12} \) and \( a_{13} \), attribute \( a_{14} \) is common attribute with membership degree 0.5.

Belohlávek has extended the fundamental theorem of concept lattices by Dedekind-MacNeille completion in fuzzy settings by using the notions of \( L \)-equality and \( L \)-ordering. All the definitions and related constructions given until the end of this section are from [9].

**Definition 5** A binary \( L \)-relation \( \approx \) on \( X \) is called an **\( L \)-equality** if it satisfies
1. \( (x \approx x) = 1 \), (reflexivity),
2. \( (x \approx y) = (y \approx x) \), (symmetry),
3. \( (x \approx y) \otimes (y \approx z) \leq (x \approx z) \), (transitivity),
4. \( (x \approx y) = 1 \) implies \( x = y \)

\( L \)-equality is a natural generalization of the classical (bivalent) notion.

**Definition 6** An **\( L \)-ordering** (or fuzzy ordering) on a set \( X \) endowed with an \( L \)-equality relation \( \approx \) is a binary \( L \)-relation \( \leq \) which is compatible w.r.t. \( \approx \) (i.e. \( f(x) \otimes (x \approx y) \leq f(y) \), for all \( x, y \in X \)) and satisfies
1. \( x \leq x = 1 \), (reflexivity),
2. \( (x \leq y) \wedge (y \leq x) \leq (x \approx y) \), (antisymmetry),
3. \( (x \leq y) \otimes (y \leq z) \leq (x \leq z) \), (transitivity).

If \( \leq \) is an \( L \)-order on a set \( X \) with an \( L \)-equality \( \approx \), we call the pair \( \langle (X, \approx) \leq \rangle \) an **\( L \)-ordered set.**
Clearly, if $L = 2$, the notion of $L$-order coincides with the usual notion of (partial) order.

**Definition 7** An $L$-set $f \in L^X$ is said to be an **$L$-singleton** in $(X, \cong)$ if it is compatible w.r.t. $\cong$ and the following holds:

1. There exists $x_0 \in X$ with $f(x_0) = 1$
2. $f(x) \otimes f(y) \leq (x \cong y)$, for all $x, y \in X$.

**Definition 8** For an $L$-ordered set $(X, \cong)$ and $f \in L^X$ the $L$-sets $\inf(f)$ and $\sup(f)$ in $X$ are defined by

1. $\inf(f)(x) = (\bigwedge_{y \in X} f(y) \to (x \leq y))$
2. $\sup(f)(x) = (\bigvee_{y \in X} f(y) \to (y \leq x))$

where

$L(f) = \bigwedge_{y \in X} (f(y) \to (x \leq y))$ and $U(f) = \bigvee_{y \in X} (f(y) \to (y \leq x))$

The $L$-sets $\inf(f)$ and $\sup(f)$ are called **infimum** and **supremum**, respectively.

**Definition 9** An $L$-ordered set $(X, \cong)$ is said to be **completely lattice $L$-ordered set** if for any $f \in L^X$ both $\inf(f)$ and $\sup(f)$ are $\cong$-singletons.

In the proof of the following chain of lemmas some well-known properties of residuated lattices are used (details can be found in [8]). Some of the needed properties are listed below.

\[(k \rightarrow (l \rightarrow m)) = ((k \otimes l) \rightarrow m) = ((l \otimes k) \rightarrow m) = (l \rightarrow (k \rightarrow m))\]  
\[k \rightarrow \bigwedge_{i \in I} m_i = \bigwedge_{i \in I} (k \rightarrow m_i)\]  
\[\bigvee_{i \in I} m_i \rightarrow k = \bigwedge_{i \in I} (m_i \rightarrow k)\]

**Lemma 1** For any pair of $L$-concepts $(f_i, g_i) \in L\text{-FCL}(B, A, r)$ ($i \in \{1, 2\}$) of any $L$-context $(B, A, r)$ the following equality holds.

\[\bigwedge_{a \in B} (f_1(a) \rightarrow f_2(a)) = \bigwedge_{a \in A} (g_2(a) \rightarrow g_1(a))\]

**Proof**

\[\bigwedge_{a \in B} (f_1(a) \rightarrow f_2(a)) = \bigwedge_{a \in B} (f_1(a) \rightarrow \downarrow (g_2(a)))\]

\[\overset{(1)}{=} \bigwedge_{a \in B} \left( f_1(a) \rightarrow \bigwedge_{a \in A} (g_2(a) \rightarrow r(a, a)) \right)\]

\[\overset{(3)}{=} \bigwedge_{a \in B} \bigwedge_{a \in A} (f_1(a) \rightarrow (g_2(a) \rightarrow r(a, a)))\]
\[
(\text{2}) \quad \bigwedge_{o \in B} \bigwedge_{a \in A} (g_2(a) \rightarrow (f_1(o) \rightarrow r(o, a)))
\]
\[
(\text{3}) \quad \bigwedge_{a \in A} \left( g_2(a) \rightarrow \bigwedge_{o \in B} (f_1(o) \rightarrow r(o, a)) \right)
\]
\[
(\text{1}) \quad \bigwedge_{a \in A} (g_2(a) \rightarrow (f_1)(a))
\]
\[
= \bigwedge_{a \in A} (g_2(a) \rightarrow g_1(a)) \quad \square
\]

**Definition 10** Let \( L \)-equality \( \approx \) and \( L \)-ordering \( \preceq \) on the set of formal concepts \( L\text{-FCL}(C) \) of \( L \)-context \( C \) are defined as follows:

\[
(f_1, g_1) \preceq (f_2, g_2) = \bigwedge_{o \in B} (f_1(o) \rightarrow f_2(o)) = \bigwedge_{a \in A} (g_2(a) \rightarrow g_1(a))
\]
\[
(\text{5})
\]
\[
(f_1, g_1) \approx (f_2, g_2) = \bigwedge_{o \in B} (f_1(o) \leftrightarrow f_2(o)) = \bigwedge_{a \in A} (g_2(a) \leftrightarrow g_1(a))
\]
\[
(\text{6})
\]

where \( k \leftrightarrow m \) is defined as \((k \rightarrow m) \land (m \rightarrow k)\) for any \( k, m \in L \).

**Example 2** In the following tables we can see the \( L \)-ordering on the sets of \( L \)-concepts of \( C_1 \) and \( C_2 \) from Example 1.

<table>
<thead>
<tr>
<th>( \preceq_1 )</th>
<th>( p_1 )</th>
<th>( p_2 )</th>
<th>( p_3 )</th>
<th>( p_4 )</th>
<th>( p_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_1 )</td>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>( p_2 )</td>
<td>1</td>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>( p_3 )</td>
<td>1</td>
<td>0.5</td>
<td>1</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>( p_4 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>( p_5 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \preceq_2 )</th>
<th>( q_1 )</th>
<th>( q_2 )</th>
<th>( q_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_1 )</td>
<td>1</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>( q_2 )</td>
<td>2</td>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>( q_3 )</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Definition 11** Let \( C = (B, A, r) \) be an \( L \)-fuzzy formal context and \( \gamma \) be an \( L \)-set from \( L^L\text{-FCL}(C) \). \( L \)-sets of objects and attributes \( \bigcup_B \gamma \) and \( \bigcup_A \gamma \) are defined as follows:

1. \((\bigcup_B \gamma)(o) = \bigvee_{(f, g) \in L^L\text{-FCL}(C)} (\gamma((f, g) \otimes f(o)))\) for \( o \in B \)
2. \((\bigcup_A \gamma)(a) = \bigvee_{(f, g) \in L^L\text{-FCL}(C)} (\gamma((f, g) \otimes g(a)))\) for \( a \in A \)

**Theorem 1** Let \( C = (B, A, r) \) be an \( L \)-context. \( \langle (L\text{-FCL}(C)), \approx, \preceq \rangle \) is a completely lattice \( L \)-ordered set in which infima and suprema can be described as follows: for an \( L \)-set \( \gamma \in L^L\text{-FCL}(C) \) we have:

\[
\inf \gamma = \{(\downarrow \bigcup_B \gamma) \uplus (\downarrow \bigcup_A \gamma)\}
\]
\[
\sup \gamma = \{(\uparrow \bigcup_B \gamma) \uplus (\uparrow \bigcup_A \gamma)\}
\]

Moreover a completely lattice \( L \)-ordered set \( V = \langle (V, \leq) \rangle \) is said to be isomorphic to \( \langle (L\text{-FCL}(C)), \approx, \preceq \rangle \) if there are mappings \( \gamma : B \times L \rightarrow V \) and \( \mu : A \times L \rightarrow V \), such that \( \gamma(B \times L) \) is \( \{0, 1\} \)-supremum dense and \( \mu(A \times L) \) is \( \{0, 1\} \)-infimum dense in \( V \), and \((k \otimes l) \rightarrow r(o, a)) = (\gamma(o, k) \preceq (\gamma(a, l))) \) for all \( o \in B \), \( a \in A \) and \( k, l \in L \). In particular, \( V \) is isomorphic to \( \langle (L\text{-FCL}(V, V, \leq), \approx, \preceq \rangle \rangle \).

If \( L = 2 \), the previous theorem coincides with the standard version of the fundamental theorem of concept lattices.
2.2 \(L\)-Isotone Galois connection

Now an \(L\)-fuzzy extension of the notion of isotone Galois connection will be introduced. Firstly we will define an \(L\)-isotone mapping between two \(L\)-ordered sets.

Definition 12 Let \(\langle V_i, \preceq_i \rangle\) for \(i \in \{1, 2\}\) be two \(L\)-ordered sets. A mapping \(s: V_1 \rightarrow V_2\) is said to be \(L\)-isotone if for any \(u, v \in V_1\) the following holds
\[
(u \preceq_1 v) \leq (s(u) \preceq_2 s(v)).
\]

It is not difficult to check that this definition extends that in the classical case.

Lemma 2 \(2\)-isotone mappings correspond to classical isotone mappings.

The definition of \(L\)-isotone Galois connection is given below:

Definition 13 Let \(\langle V_i, \approx_i \rangle\) for \(i \in \{1, 2\}\) be two \(L\)-ordered sets. An \(L\)-isotone Galois connection is a pair of \(L\)-isotone mappings \((s, z)\) such that \(s: V_1 \rightarrow V_2\) and \(z: V_2 \rightarrow V_1\) and for any pair \((v_1, v_2) \in V_1 \times V_2\) the following equality holds
\[
(s(v_1) \preceq_2 v_2) = (v_1 \preceq_1 z(v_2)).
\]

Lemma 3 The \(2\)-isotone Galois connections correspond to classical isotone Galois connections.

Proof From the previous lemma, we know that if \(L = 2\) then \(s\) and \(z\) are classical isotone mappings between ordered sets \((V_1, \leq_1)\) and \((V_2, \leq_2)\). Equality of values \((s(v_1) \preceq_2 v_2)\) and \((v_1 \preceq_1 z(v_2))\) that are from \{0, 1\} makes the equivalence \((s(v_1) \preceq_2 v_2) \Leftrightarrow (v_1 \preceq_1 z(v_2))\) hold true. Hence \((s, z)\) forms an isotone Galois connection. \(\square\)

2.3 \(L\)-Chu correspondences

For the sake of self-containment, in this section we recall the main definitions concerning \(L\)-Chu correspondences, which were already used in our previous works [27, 28].

Definition 14 An \(L\)-multifunction from set \(X\) to set \(Y\) is a mapping from \(X\) to \(L^Y\).

Definition 15 Given \(\varpi: X \rightarrow L^Y\), a mapping \(\varpi_+: L^X \rightarrow L^Y\) for all \(f \in L^X\) is defined by
\[
\varpi_+(f)(y) = \bigvee_{x \in X} (f(x) \otimes \varpi(x)(y)).
\]

Definition 16 Consider two \(L\)-fuzzy contexts \(C_i = (B_i, A_i, r_i), (i = 1, 2)\), then the pair \(\varphi = (\varphi_L, \varphi_R)\) is said to be a correspondence from \(C_1\) to \(C_2\) if \(\varphi_L\) and \(\varphi_R\) are \(L\)-multifunctions, respectively, from \(B_1\) to \(B_2\) and from \(A_2\) to \(A_1\) (ie, \(\varphi_L: B_1 \rightarrow L^{B_2}\) and \(\varphi_R: A_2 \rightarrow L^{A_1}\)).

The \(L\)-correspondence \(\varphi\) is said to be a weak \(L\)-Chu correspondence if the following equality holds for all \(o_1 \in B_1\) and \(o_2 \in B_2\):
\[
\bigwedge_{o_1 \in A_1} (\varphi_R(o_2)(a_1) \rightarrow r_1(o_1, a_1)) = \bigwedge_{o_2 \in B_2} (\varphi_L(o_1)(o_2) \rightarrow r_2(o_2, a_2)).
\]

A weak Chu correspondence \(\varphi\) is an \(L\)-Chu correspondence if \(\varphi_L(o_1)\) is an \(L\)-set of objects closed in \(C_2\) and \(\varphi_R(o_2)\) is an \(L\)-set of attributes closed in \(C_1\) for all \(o_1 \in B_1\) and \(o_2 \in A_2\). We will denote the set of all \(L\)-Chu correspondences from \(C_1\) to \(C_2\) by \(L\text{-ChuCors}(C_1, C_2)\).
Table 1

<table>
<thead>
<tr>
<th></th>
<th>(-)(_L)</th>
<th>(-)(_N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\varphi_1)</td>
<td>(\begin{array}{ccc} o_{21} &amp; o_{22} &amp; o_{23} \ o_{11} &amp; 1 &amp; 1 &amp; 0.5 \ o_{12} &amp; 1 &amp; 1 &amp; 0.5 \ o_{21} &amp; 1 &amp; 0.5 &amp; 0.5 \ o_{22} &amp; 1 &amp; 0.5 &amp; 0.5 \ o_{23} &amp; 1 &amp; 0.5 &amp; 0.5 \end{array})</td>
<td>(\begin{array}{ccc} a_{11} &amp; a_{12} &amp; a_{13} &amp; a_{14} \ o_{21} &amp; 1 &amp; 1 &amp; 1 &amp; 0.5 \ o_{22} &amp; 1 &amp; 0.5 &amp; 0.5 &amp; 0 \ o_{23} &amp; 1 &amp; 0.5 &amp; 0.5 &amp; 0 \end{array})</td>
</tr>
<tr>
<td>(\varphi_2)</td>
<td>(\begin{array}{ccc} o_{21} &amp; o_{22} &amp; o_{23} \ o_{11} &amp; 1 &amp; 1 &amp; 0.5 \ o_{12} &amp; 1 &amp; 0.5 &amp; 0 \end{array})</td>
<td>(\begin{array}{ccc} a_{11} &amp; a_{12} &amp; a_{13} &amp; a_{14} \ o_{21} &amp; 1 &amp; 1 &amp; 0.5 &amp; 0 \ o_{22} &amp; 1 &amp; 0.5 &amp; 0 &amp; 0 \ o_{23} &amp; 1 &amp; 0.5 &amp; 0 &amp; 0 \end{array})</td>
</tr>
<tr>
<td>(\varphi_3)</td>
<td>(\begin{array}{ccc} o_{21} &amp; o_{22} &amp; o_{23} \ o_{11} &amp; 1 &amp; 0.5 &amp; 0 \end{array})</td>
<td>(\begin{array}{ccc} a_{11} &amp; a_{12} &amp; a_{13} &amp; a_{14} \ o_{21} &amp; 1 &amp; 1 &amp; 0.5 &amp; 0 \ o_{22} &amp; 1 &amp; 0.5 &amp; 0 &amp; 0 \ o_{23} &amp; 1 &amp; 0.5 &amp; 0 &amp; 0 \end{array})</td>
</tr>
<tr>
<td>(\varphi_4)</td>
<td>(\begin{array}{ccc} o_{21} &amp; o_{22} &amp; o_{23} \ o_{11} &amp; 1 &amp; 0.5 &amp; 0 \end{array})</td>
<td>(\begin{array}{ccc} a_{11} &amp; a_{12} &amp; a_{13} &amp; a_{14} \ o_{21} &amp; 1 &amp; 1 &amp; 1 \ o_{22} &amp; 1 &amp; 0.5 &amp; 0.5 \ o_{23} &amp; 1 &amp; 0.5 &amp; 0.5 \end{array})</td>
</tr>
<tr>
<td>(\varphi_5)</td>
<td>(\begin{array}{ccc} o_{21} &amp; o_{22} &amp; o_{23} \ o_{11} &amp; 1 &amp; 1 \end{array})</td>
<td>(\begin{array}{ccc} a_{11} &amp; a_{12} &amp; a_{13} &amp; a_{14} \ o_{21} &amp; 1 &amp; 1 &amp; 0 \end{array})</td>
</tr>
</tbody>
</table>

Example 3 All \(L\)-Chu correspondences between the \(L\)-contexts \(C_1\) and \(C_2\) used in Example 1 can be seen in Table 1.

In the left column of the table one can see all the left parts \(\varphi_L\), which are \(L\)-multifunctions that assign some extent of \(C_2\) to every object of \(C_1\). In the right column of the table the corresponding right parts of the \(L\)-Chu correspondences \(\varphi_R\) are shown; they assign some intent of \(C_1\) to every attribute of \(C_2\) in such a way that equality (8) from Definition 16 holds.

3 Introducing the relevant categories

3.1 The category \(L\)-ChuCors

The category of \(L\)-fuzzy formal contexts and \(L\)-Chu correspondences between them is formally defined below:

- **objects** \(L\)-fuzzy formal contexts
- **arrows** \(L\)-Chu correspondences
- **identity arrow** \(\iota : C \rightarrow C\) of \(L\)-context \(C = (B, A, r)\)
  - \(\iota_L(o) = \uparrow (\chi_o)\), for all \(o \in B\)
  - \(\iota_R(a) = \downarrow (\chi_a)\), for all \(a \in A\)
  where \(\chi_x(x) = 1\) and \(\chi_x(y) = 0\) for any \(y \neq x\)
- **composition** \(\varphi_2 \circ \varphi_1 : C_1 \rightarrow C_3\) of arrows \(\varphi_1 : C_1 \rightarrow C_2\), \(\varphi_2 : C_2 \rightarrow C_3\) \((C_i = (B_i, A_i, r_i), i \in \{1, 2\})\)
\(- (\varphi_2 \circ \varphi_1)_L : B_1 \rightarrow L^{B_3}\) defined as
\[
(\varphi_2 \circ \varphi_1)_L(o_1) = \downarrow_3 \uparrow_3 (\varphi_{2L+}(\varphi_{1L}(o_1)))
\]  
(9)
where
\[
\varphi_{2L+}(\varphi_{1L}(o_1))(o_3) = \bigvee_{o_2 \in B_2} \varphi_{1L}(o_1)(o_2) \otimes \varphi_{2L}(o_2)(o_3)
\]

\(- (\varphi_2 \circ \varphi_1)_R : A_3 \rightarrow L^{A_1}\) defined as
\[
(\varphi_2 \circ \varphi_1)_R(a_3) = \downarrow_1 \uparrow_1 (\varphi_{1R+}(\varphi_{2R}(a_3)))
\]  
(10)
where
\[
\varphi_{1R+}(\varphi_{2R}(a_3))(a_1) = \bigvee_{a_2 \in A_2} \varphi_{2R}(a_3)(a_2) \otimes \varphi_{1R}(a_2)(a_1)
\]

All details about the definition of the category \(L\)-ChuCors could be found in [27].

3.2 Category \(L\)-CLOS

Here we define another category

**Objects** are completely lattice \(L\)-ordered sets \((L\text{-CLOS})\) i.e. our objects will be represented as \(V = (V, \approx, \succeq)\)

**Arrows** are \(L\)-isotone Galois connections between two \(L\)-CLOS i.e. \((s, z)\) between \(V_1\) and \(V_2\), such that:

1. \(s : V_1 \rightarrow V_2\),
2. \(z : V_2 \rightarrow V_1\),
3. \((s(v_1) \preceq z(v_2)) = (v_1 \preceq z(v_2))\) for all \((v_1, v_2) \in V_1 \times V_2\).

**Identity arrow** of \((V, \approx, \succeq)\) is a pair of identity morphisms \((id_V, id_V)\)

**Composition of arrows** is based on composition of mappings: consider two arrows \((s_i, z_i) : V_i \rightarrow V_{i+1}\), where \(i \in \{1, 2\}\). Composition is defined as follows:
\[
(s_2 \circ s_1, z_2 \circ z_1)
\]

Thus, given a pair of two arbitrary elements \((v_1, v_3) \in V_1 \times V_3\) then:

\[
((s_2 \circ s_1)(v_1) \preceq z_3 v_3) = (s_2(s_1(v_1)) \preceq z_3 v_3)
\]
\[
= (s_1(v_1) \preceq z_2(z_3(v_3)))
\]
\[
= (v_1 \preceq z_1(z_2(v_3)))
\]
\[
= (v_1 \preceq z_1(z_2 \circ z_3)(v_3))
\]

**Associativity of composition** follows trivially because of the associativity of composition of mappings between sets.
4 The categories $L$-ChuCors and $L$-CLOS are equivalent

As stated in [4], mathematically significant properties of objects are those that are invariant under isomorphisms and, in category theory, equivalence of categories is the most convenient notion of “isomorphism” (used here with an informal meaning) between categories.

In this section, we reach the main goal of this paper: to prove the equivalence of the categories $L$-ChuCors and $L$-CLOS. As a result, we obtain that the generalized approaches based on $L$-concept lattices of $C_1$ and $C_2$ are mutually interchangeable.

The equivalence between both categories will be proved by defining a suitable functor $\Gamma$ which links $L$-ChuCors to $L$-CLOS. The behavior of $\Gamma$ for objects is straightforward: to any $L$-context $C$ the functor $\Gamma$ assigns its corresponding concept $L$-CLOS, namely $L$-FCL$(C)$. The formal definition is given below:

1. $\Gamma(C) = \langle (L$-FCL$(C), \leq) \rangle$
2. For any $L$-Chu correspondence $\varphi \in L$-ChuCors$(C_1, C_2)$, the result of $\Gamma(\varphi)$ will be a pair of mappings $\langle \varphi_\vee, \varphi_\wedge \rangle$ defined as follows:

$$\varphi_\vee \left((f_i,g_i)\right) = \langle i_2 \varphi_{L_+}(f_i), i_2 \varphi_{L_+}(f_i) \rangle \quad (11)$$
$$\varphi_\wedge \left((f_i,g_i)\right) = \langle i_1 \varphi_{R_+}(g_i), i_1 \varphi_{R_+}(g_i) \rangle \quad (12)$$

where $(f_i,g_i) \in L$-FCL$(C_i)$ for $i \in \{1,2\}$.

Example 4 Continuing with our running example, Table 2 contains all pairs of mappings $\langle \varphi_\vee, \varphi_\wedge \rangle$ between $L$-concept lattices of $C_1$ and $C_2$ that are assigned by the mapping $\Gamma$ to all $L$-Chu correspondences $\varphi$ between $L$-contexts $C_1$ and $C_2$ from Example 3.

<table>
<thead>
<tr>
<th>$\varphi_\vee$</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$P_4$</th>
<th>$P_5$</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$q_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi_\vee(-)$</td>
<td>$q_2$</td>
<td>$q_2$</td>
<td>$q_2$</td>
<td>$q_3$</td>
<td>$q_3$</td>
<td>$\varphi_\vee(-)$</td>
<td>$P_1$</td>
<td>$P_1$</td>
</tr>
<tr>
<td>$\varphi_\vee(\neg)$</td>
<td>$q_2$</td>
<td>$q_2$</td>
<td>$q_3$</td>
<td>$q_3$</td>
<td>$q_3$</td>
<td>$\varphi_\vee(\neg)$</td>
<td>$P_1$</td>
<td>$P_1$</td>
</tr>
<tr>
<td>$\varphi_\wedge(\neg)$</td>
<td>$q_2$</td>
<td>$q_1$</td>
<td>$q_2$</td>
<td>$q_3$</td>
<td>$q_3$</td>
<td>$\varphi_\wedge(\neg)$</td>
<td>$P_1$</td>
<td>$P_1$</td>
</tr>
<tr>
<td>$\varphi_\wedge(-)$</td>
<td>$q_3$</td>
<td>$q_3$</td>
<td>$q_3$</td>
<td>$q_3$</td>
<td>$q_3$</td>
<td>$\varphi_\wedge(-)$</td>
<td>$P_1$</td>
<td>$P_1$</td>
</tr>
</tbody>
</table>

The two following lemmas are needed in order to prove that the morphism part of the functor $\Gamma$ is well defined, that is, the pair $\langle \varphi_\vee, \varphi_\wedge \rangle$ is an $L$-isotone Galois connection between $\Gamma(C_1)$ and $\Gamma(C_2)$.

Lemma 4 Let $C = \langle B, A, r \rangle$ be an $L$-context. For any $L$-set $f \in L^B$ and any concept $\langle h, g \rangle$ holds:

$$\bigwedge_{o \in B} \langle \uparrow f \downarrow(\langle f(\langle o \rangle \rightarrow h(\langle o \rangle) \rangle) = \bigwedge_{o \in B} \langle f(\langle o \rangle \rightarrow h(\langle o \rangle) \rangle$$
Lemma 5 Let \( C_i = \langle B_i, A_i, r_i \rangle \) for \( i \in \{1, 2\} \) be two arbitrary \( L \)-contexts. Let \( \omega \) be an \( L \)-multifunction between \( B_1 \) and \( B_2 \) (\( \omega : B_1 \rightarrow L^{B_2} \)) and \( f \) be an arbitrary \( L \)-set from \( L^{B_1} \). Then
\[
\uparrow_2 (\omega_+(f))(a) = \bigwedge_{b \in B_1} (f(b) \rightarrow \uparrow_2 (\omega(b))(a)).
\]

Proof
\[
\uparrow_2 (\omega_+(f))(a) \overset{(1)}{=} \bigwedge_{o \in B_2} (\omega_+(f)(o) \rightarrow r_2(o, a))
\]
\[
\overset{(7)}{=} \bigwedge_{o \in B_2} \left( \bigvee_{b \in B_1} (f(b) \otimes \omega(b)(o)) \rightarrow r_2(o, a) \right)
\]
\[
\overset{(4)}{=} \bigwedge_{o \in B_2} \bigwedge_{b \in B_1} ((f(b) \otimes \omega(b)(o)) \rightarrow r_2(o, a))
\]
\[
\overset{(2)}{=} \bigwedge_{o \in B_2} \bigwedge_{b \in B_1} (f(b) \rightarrow (\omega(b)(o)) \rightarrow r_2(o, a))
\]
\[
\begin{align*}
\sum_{b \in B_1} \left( f(b) \rightarrow \sum_{a \in B_2} (\omega(b)(a)) \rightarrow r_2(o, a) \right) \\
\sum_{b \in B_1} (f(b) \rightarrow \Gamma_2(\omega(b))(a))
\end{align*}
\]

With the help of the previous lemmas we can now prove that the morphism part of \(\Gamma\) is well-defined.

**Lemma 6** \(\Gamma(\varphi) \in L\text{-CLLOS}(\Gamma(C_1), \Gamma(C_2))\) for any \(\varphi \in L\text{-ChuCors}(C_1, C_2)\).

**Proof** Firstly, \(L\)-isotonicity of the pair of mappings \(\langle \varphi_V, \varphi_\wedge \rangle\) will be shown. Let us consider \(\langle f, \Gamma_1(f) \rangle\) and \(\langle h, \Gamma_1(h) \rangle\) be two \(L\)-concepts of context \(C_1 = (B_1, A_1, r_1)\)

\[\varphi_V((f, \Gamma_1(f))) \leq \varphi_V((h, \Gamma_1(h)))\]

\[\sum_{o \in B_2} (\Gamma_2(\varphi_L+(f))(o) \rightarrow \Gamma_2(\varphi_L+(h))(o))\]

because of Lemma 4

\[\sum_{o \in B_2} (\varphi_L+(f)(o) \rightarrow \Gamma_2(\varphi_L+(h))(o))\]

\[\sum_{o \in B_2} \left( \bigvee_{b \in B_1} (\varphi_L(b)(o) \otimes f(b)) \rightarrow \Gamma_2(\varphi_L+(h))(o) \right)\]

\[\sum_{o \in B_2} \left( \bigwedge_{b \in B_1} ((\varphi_L(b)(o) \otimes f()) \rightarrow \Gamma_2(\varphi_L+(h))(o)) \right)\]

\[\sum_{o \in B_2} \left( \bigwedge_{b \in B_1} (f(b) \rightarrow (\varphi_L(b)(o) \rightarrow \Gamma_2(\varphi_L+(h))(o))) \right)\]

\[\sum_{o \in B_2} \left( \bigvee_{b \in B_1} (\varphi_L(b)(o) \rightarrow \Gamma_2(\varphi_L+(h))(o)) \right)\]

because of Lemma 1

\[\sum_{b \in B_1} (f(b) \rightarrow \sum_{a \in A_2} (\Gamma_2(\varphi_L+(h))(a) \rightarrow \Gamma_2(\varphi_L(b))(a)))\]

because of Lemma 5

\[\sum_{b \in B_1} \left( f(b) \rightarrow \sum_{a \in A_2} \left( \bigwedge_{j \in B_1} (\beta(j) \rightarrow \Gamma_2(\varphi_L(j))(a)) \rightarrow \Gamma_2(\varphi_L(b))(a) \right) \right)\]

denoting \(\Gamma_2(\varphi_L(b))(a) = \beta(b, a)\)

\[\sum_{b \in B_1} \left( f(b) \rightarrow \sum_{a \in A_2} \left( \bigwedge_{j \in B_1} (\beta(j) \rightarrow \beta(j, a)) \rightarrow \beta(b, a) \right) \right)\]
\[\begin{align*}
\text{(6)} & \quad \bigwedge_{b \in B_1} (f(b) \rightarrow \bot_b \bot_b (h)(b)) \\
\text{by the property of closure} \\
\geq & \quad \bigwedge_{b \in B_1} (f(b) \rightarrow h(b)) \\
= & \quad (\langle f, \bot_1 (f) \rangle \leq_1 \langle h, \bot_1 (h) \rangle)
\end{align*}\]

Hence \( \varphi \vee \) is \( L \)-isotone. Similarly for \( \varphi \wedge \).

Consider two arbitrary \( L \)-concepts \( \langle f_i, g_i \rangle \) of \( \langle (L, \text{FCL}(C_i), \approx_i), \leq_i \rangle \) for \( i \in \{1, 2\} \) where \( C_i = (B_i, \alpha_i, r_i) \).

\[
\varphi(\langle f_1, g_1 \rangle) \leq_2 \langle f_2, g_2 \rangle \quad \text{(11)}
\]

\[
\begin{align*}
\text{(5)} & \quad \bigwedge_{a_2 \in A_2} (g_2(a_2) \rightarrow \langle \varphi_{L+}(f_1)(a_2) \rangle)
\text{(1)} & \quad \bigwedge_{a_2 \in A_2} \left( g_2(a_2) \rightarrow \bigwedge_{a_2 \in B_2} \left( \varphi_{L+}(f_1)(a_2) \rightarrow r_2(a_2, a_2) \right) \right)
\text{(7)} & \quad \bigwedge_{a_2 \in A_2} \left( g_2(a_2) \rightarrow \bigwedge_{a_2 \in B_2} \left( \bigvee_{a_2 \in B_1} \left( \varphi_{L}(a_1)(a_2) \otimes f_1(a_1) \right) \rightarrow r_2(a_2, a_2) \right) \right)
\text{(4)} & \quad \bigwedge_{a_2 \in A_2} \left( g_2(a_2) \rightarrow \bigwedge_{a_2 \in B_2} \left( \varphi_{L}(a_1)(a_2) \otimes f_1(a_1) \rightarrow r_2(a_2, a_2) \right) \right)
\text{(3)} & \quad \bigwedge_{a_2 \in A_2} \left( g_2(a_2) \rightarrow \left( \varphi_{L}(a_1)(a_2) \otimes f_1(a_1) \rightarrow r_2(a_2, a_2) \right) \right)
\text{(2)} & \quad \bigwedge_{a_2 \in A_2} \left( g_2(a_2) \rightarrow \left( \varphi_{L}(a_1)(a_2) \rightarrow f_1(a_1) \rightarrow r_2(a_2, a_2) \right) \right)
\text{(1)} & \quad \bigwedge_{a_2 \in A_2} \left( g_2(a_2) \rightarrow \left( f_1(a_1) \rightarrow \bigwedge_{a_2 \in B_2} \left( \varphi_{L}(a_1)(a_2) \rightarrow r_2(a_2, a_2) \right) \right) \right)
\text{(8)} & \quad \bigwedge_{a_2 \in A_2} \left( f_1(a_1) \rightarrow \left( g_2(a_2) \rightarrow \bigwedge_{a_2 \in A_1} \left( \varphi_{R}(a_1)(a_1) \rightarrow r_1(a_1, a_1) \right) \right) \right)
\text{by a similar chain of computation we obtain}
\end{align*}\]

\[
\begin{align*}
= & \quad \bigwedge_{a_2 \in A_2} \left( f_1(a_1) \rightarrow \bigwedge_{a_2 \in A_2} \left( \bigvee_{a_2 \in A_1} \left( \varphi_{R}(a_2)(a_1) \otimes g_2(a_2) \right) \rightarrow r_1(a_1, a_1) \right) \right)
\text{(7)} & \quad \bigwedge_{a_2 \in A_2} \left( f_1(a_1) \rightarrow \bot_1 \left( \varphi_{R+}(g_2)(a_1) \right) \right)
\text{(5)} & \quad \left( \langle f_1, g_1 \rangle \leq_1 \langle 1 \bot_1 (\varphi_{R+}(g_2)), 1 \bot_1 (\varphi_{R+}(g_2)) \rangle \right)
\text{(12)} & \quad \left( \langle f_1, g_1 \rangle \leq_1 \varphi \wedge \langle f_2, g_2 \rangle \right)
\end{align*}\]

So \( \langle \varphi \vee, \varphi \wedge \rangle \) is an \( L \)-isotone Galois connection between the completely lattice \( L \)-ordered sets \( \langle (L, \text{FCL}(C_1), \approx_1), \leq_1 \rangle \) and \( \langle (L, \text{FCL}(C_2), \approx_2), \leq_2 \rangle \). \( \Box \)
Example 5 In Table 3 one can see that all pairs \( \langle \varphi_v, \varphi_\lambda \rangle \) in Example 4 with the \( L \)-ordering given in Example 2 satisfy Definition 13 and, therefore, all are \( L \)-isotone Galois connections.

The following result checks that \( \Gamma \) preserves identity morphisms.

**Lemma 7** For the identity arrow \( \iota \in \text{L-ChuCors}(C, C) \) of any \( L \)-context \( C = (B, A, r) \), \( \Gamma(\iota) \) is the identity arrow from \( \text{L-CLOS}(F(C), F(C)) \).

**Proof** Consider any \( L \)-concept \( (f, g) \) from \( L\text{-FCL}(C) \).

\[
\uparrow (i_{L+}(f))(a) = \bigwedge_{o \in B} (i_{L+}(f)(o) \rightarrow r(o, a))
\]
Consider two arbitrary \( \Gamma \) and \( \iota \). Therefore, we have \( \iota \) preserves composition. Therefore, \( \iota \) is similar.

We continue with a technical lemma which proves an equality needed in the proof that \( \Gamma \) preserves composition.

**Lemma 8**  Consider two arbitrary \( \varphi_i \in L\text{-ChuCor}(C_i,C_{i+1}) \) for \( i \in \{1,2\} \) and any element \( a_1 \in B_1 \) and \( g_3 \in L^{A_3} \). Then

\[
\downarrow \left( \varphi_{1R} + (\varphi_{2R} + (g_3)) \right)(a_1) = \downarrow \left( \varphi_{1R} + (\downarrow \downarrow (\varphi_{2R} + (g_3))) \right)(a_1).
\]

**Proof**

\[
\downarrow \left( \varphi_{1R} + (\varphi_{2R} + (g_3)) \right)(a_1) = \\
\downarrow \left( \varphi_{1R} + (\downarrow \downarrow (\varphi_{2R} + (g_3))) \right)(a_1).
\]

\[
\downarrow \left( \varphi_{1R} + (\varphi_{2R} + (g_3)) \right)(a_1) = \\
\downarrow \left( \varphi_{1R} + (\downarrow \downarrow (\varphi_{2R} + (g_3))) \right)(a_1).
\]

\[
\downarrow \left( \varphi_{1R} + (\varphi_{2R} + (g_3)) \right)(a_1) = \\
\downarrow \left( \varphi_{1R} + (\downarrow \downarrow (\varphi_{2R} + (g_3))) \right)(a_1).
\]

\[
\downarrow \left( \varphi_{1R} + (\varphi_{2R} + (g_3)) \right)(a_1) = \\
\downarrow \left( \varphi_{1R} + (\downarrow \downarrow (\varphi_{2R} + (g_3))) \right)(a_1).
\]

by the closure property

\[
\downarrow \left( \varphi_{1R} + (\varphi_{2R} + (g_3)) \right)(a_1) = \\
\downarrow \left( \varphi_{1R} + (\downarrow \downarrow (\varphi_{2R} + (g_3))) \right)(a_1).
\]
Finally, by applying the same chain of modifications in opposite way we will obtain

\[ \downarrow_1 (\varphi_1 R+ (\uparrow_2 R_{2R}(g_3))) (a_1) \]

\( \square \)

**Lemma 9** \( \Gamma \) is closed under arrow composition.

**Proof** Consider \( \varphi_i \in L\text{-ChuCors}(C_i, C_{i+1}) \) for \( i \in \{1, 2\} \). Let \( \langle f_i, g_i \rangle \in L\text{-FCL}(C_i) \) be an arbitrary \( L \)-context for all \( i \in \{1, 3\} \). Recall that

1. \( \Gamma(\varphi_2 \circ \varphi_1) = \langle (\varphi_2 \circ \varphi_1)_v, (\varphi_2 \circ \varphi_1)_\wedge \rangle \)
2. \( \Gamma(\varphi_2) \circ \Gamma(\varphi_1) = \langle \varphi_2 \circ \varphi_1_v, \varphi_1 \circ \varphi_2 \rangle \)

The proof will be based on equality of corresponding elements of the previous pairs: only one part will be proved, the other one is similar.

\[
\downarrow_1 (\varphi_1 R+ (\varphi_2 R+ (g_3))) (a_1) = \\
\downarrow_1 (\varphi_1 R+ (\varphi_2 R+ (g_3))) (a_1) =
\downarrow_1 \bigwedge_{a_1 \in A_1} \bigg( \bigvee_{a_2 \in A_2} \bigg( \varphi_1 R(a_2)(a_1) \otimes \varphi_2 R(a_2)(a_2) \to r_1(a_1, a_1) \bigg) \bigg)
\downarrow_1 \bigwedge_{a_1 \in A_1} \bigg( \bigvee_{a_2 \in A_2} \bigg( \varphi_1 R(a_2)(a_1) \otimes \bigvee_{a_3 \in A_3} \varphi_2 R(a_3)(a_2) \otimes g_3(a_3) \to r_1(a_1, a_1) \bigg) \bigg)
\downarrow_1 \bigwedge_{a_1 \in A_1} \bigg( \bigvee_{a_2 \in A_2} \bigg( \varphi_1 R+(\varphi_2 R)+(g_3)(a_1) \otimes g_3(a_3) \to r_1(a_1, a_1) \bigg) \bigg)
\downarrow_1 \bigwedge_{a_2 \in A_2} \bigg( \bigvee_{a_3 \in A_3} \bigg( g_3(a_3) \to \bigwedge_{a_1 \in A_1} \bigg( \varphi_1 R+(\varphi_2 R)+(g_3)) \to r_1(a_1, a_1) \bigg) \bigg)
\downarrow_1 \bigwedge_{a_3 \in A_3} \bigg( g_3(a_3) \to \bigwedge_{a_1 \in A_1} \bigg( \varphi_1 R+(\varphi_2 R)+(g_3)) \to r_1(a_1, a_1) \bigg)
\downarrow_1 \bigwedge_{a_1 \in A_1} \bigg( (\varphi_2 \circ \varphi_1)_R(a_3)(a_1) \to r_1(a_1, a_1) \bigg)
\downarrow_1 \bigwedge_{a_1 \in A_1} \bigg( \bigvee_{a_3 \in A_3} \bigg( (\varphi_3 \circ \varphi_1)_R(a_3)(a_1) \to r_1(a_1, a_1) \bigg) \bigg)
\downarrow_1 \bigwedge_{a_1 \in A_1} \bigg( \bigvee_{a_2 \in A_2} \bigg( g_3(a_3) \otimes (\varphi_2 \circ \varphi_1)_R(a_3)(a_1) \to r_1(a_1, a_1) \bigg) \bigg)
\downarrow_1 \bigwedge_{a_1 \in A_1} \bigg( \bigvee_{a_2 \in A_2} \bigg( g_3(a_3) \otimes (\varphi_2 \circ \varphi_1)_R(a_3)(a_1) \to r_1(a_1, a_1) \bigg) \bigg)
\downarrow_1 \bigwedge_{a_2 \in A_2} \bigg( \bigvee_{a_3 \in A_3} \bigg( g_3(a_3) \otimes (\varphi_2 \circ \varphi_1)_R(a_3)(a_1) \to r_1(a_1, a_1) \bigg) \bigg)
\downarrow_1 (\varphi_2 \circ \varphi_1)_R(a_3)(a_1) \to r_1(a_1, a_1) \bigg) \bigg)
\downarrow_1 (\varphi_2 \circ \varphi_1)_R(a_3)(a_1) \to r_1(a_1, a_1) \bigg) \bigg)
\downarrow_1 \bigwedge_{a_2 \in A_2} \bigg( \bigvee_{a_3 \in A_3} \bigg( g_3(a_3) \otimes (\varphi_2 \circ \varphi_1)_R(a_3)(a_1) \to r_1(a_1, a_1) \bigg) \bigg)
Hence
\[
(f_3, g_3)
\]
by lemma 8 we have
\[
\]
\[ \uparrow_2 \left( \varphi_{s,s}^{(s_2)}(a_1) \right)(a_2) = \bigwedge_{a_2 \in B_2} \left( \varphi_{s,s}^{(s_2)}(a_1) \rightarrow r_2(a_2, a_2) \right) \]

\[ = \bigwedge_{a_2 \in B_2} \left( \text{Ext}(s(\uparrow_1 \downarrow_1 (\chi_{a_1}), \uparrow_1 \downarrow_2 (\chi_{a_1}))(a_2) \rightarrow \uparrow_2 (\chi_{a_2})(a_2) \right) \]

\[ = s(\langle \uparrow_1 \downarrow_1 (\chi_{a_1}), \uparrow_1 \downarrow_2 (\chi_{a_1}) \rangle) \leq 2 \langle \uparrow_2 (\chi_{a_2}), \uparrow_2 \downarrow 2 (\chi_{a_2}) \rangle \]

\[ = \langle \downarrow_1 \downarrow_1 (\chi_{a_1}), \downarrow_1 \downarrow_2 (\chi_{a_1}) \rangle \leq 2 \langle \downarrow_2 (\chi_{a_2}), \downarrow_2 \downarrow 2 (\chi_{a_2}) \rangle \]

\[ = \bigwedge_{a_1 \in A_1} \left( \text{Int}(z(\langle \downarrow_2 (\chi_{a_2}), \downarrow_2 \downarrow 2 (\chi_{a_2}) \rangle))(a_1) \rightarrow \uparrow_1 (\chi_{a_1})(a_1) \right) \]

\[ = \bigwedge_{a_1 \in A_1} \left( \varphi_{s,s}^{(s_2)}(a_1) \rightarrow r_1(o_1, a_1) \right) \]

\[ = \downarrow_1 \left( \varphi_{s,s}^{(s_2)}(a_2) \right)(a_1) \]

So \( \varphi_{s,s} \in L-\text{ChuCors}(C_1, C_2) \) and \( \Gamma_{C_1, C_2} \) is surjective, hence \( \Gamma \) is full. \( \square \)

Lemma 11 \( \Gamma \) is faithful.

Proof Now the point is to prove the injectivity of \( \Gamma_{C_1, C_2} \).

Consider two \( L \)-Chu correspondences \( \varphi_1, \varphi_2 \) from \( L-\text{ChuCors}(C_1, C_2) \) such that \( \varphi_1 \neq \varphi_2 \), and let us fix the pair \((a_1, a_2) \in B_1 \times A_2\), such that

\[ \uparrow_2 \left( \varphi_{1L}(o_1) \right)(a_2) = \downarrow_1 \left( \varphi_{1R}(o_2) \right)(a_1) \neq \uparrow_2 \left( \varphi_{2L}(o_1) \right)(a_2) = \downarrow_1 \left( \varphi_{2R}(o_2) \right)(a_1) \]

Let us assume that either \( \downarrow_1 \left( \varphi_{1R}(o_2) \right)(a_1) > \uparrow_2 \left( \varphi_{2L}(o_1) \right)(a_2) \) or both values from \( L \) are incomparable, that is equivalent to the following:

\[ \downarrow_1 \left( \varphi_{1R}(o_2) \right)(a_1) \rightarrow \uparrow_2 \left( \varphi_{2L}(o_1) \right)(a_2) < 1 \]

Now consider the \( L \)-concept \( \langle \downarrow_1 \left( \varphi_{1R}(o_2) \right), \varphi_{1R}(o_2) \rangle \) and let us compare its images under the mappings \( \varphi_{1V} \) and \( \varphi_{2V} \).

\[ \uparrow_2 \left( \varphi_{2L+} \left( \downarrow_1 \left( \varphi_{1R}(o_2) \right) \right) \right)(a_2) \]

\[ \overset{(1)}{=} \bigwedge_{a_2 \in B_2} \left( \varphi_{2L+} \left( \downarrow_1 \left( \varphi_{1R}(o_2) \right) \right)(a_2) \rightarrow r_2(o_2, a_2) \right) \]

\[ \overset{(7)}{=} \bigwedge_{a_2 \in B_2} \left( \bigvee_{b_1 \in B_1} \left( \varphi_{2L}(b_1)(a_2) \uplus \downarrow_1 \left( \varphi_{1R}(o_2) \right)(b_1) \right) \rightarrow r_2(o_2, a_2) \right) \]

\[ \overset{(4)(2)(3)}{=} \bigwedge_{b_1 \in B_1} \left( \downarrow_1 \left( \varphi_{1R}(o_2) \right)(b_1) \rightarrow \bigwedge_{a_2 \in B_2} \left( \varphi_{2L}(b_1)(a_2) \rightarrow r_2(o_2, a_2) \right) \right) \]

\[ \overset{(1)}{=} \bigwedge_{b_1 \in B_1} \left( \downarrow_1 \left( \varphi_{1R}(o_2) \right)(b_1) \rightarrow \uparrow_2 \left( \varphi_{2L}(b_1) \right)(a_2) \right) \]

\[ \leq \downarrow_1 \left( \varphi_{1R}(o_2) \right)(a_1) \rightarrow \uparrow_2 \left( \varphi_{2L}(o_1) \right)(a_2) \]

because of the restriction given above

\[ < 1 \]
Similarly, we can obtain:

\[
\uparrow^2(\varphi_{1L}^+(\downarrow_1(\varphi_{1R}(a_2))))(a_2) = \bigwedge_{b_1 \in B_1} (\downarrow_1(\varphi_{1R}(a_2))(b_1) \rightarrow \uparrow^2(\varphi_{1L}(b_1))(a_2)) \overset{(8)}{=} \bigwedge_{b_1 \in B_1} (\downarrow_1(\varphi_{1R}(a_2))(b_1) \rightarrow \downarrow_1(\varphi_{1R}(a_2))(b_1)) = 1
\]

It means that \(\varphi_{1V}(\downarrow_1(\varphi_{1R}(a_2)), \varphi_{1R}(a_2)) \neq \varphi_{2V}(\downarrow_1(\varphi_{1R}(a_2)), \varphi_{1R}(a_2))\)

Hence \(\varphi_{1V}(\downarrow_1(\varphi_{1R}(a_2)), \varphi_{1R}(a_2)) \neq \varphi_{2V}(\downarrow_1(\varphi_{1R}(a_2)), \varphi_{1R}(a_2))\) and \(\varphi_{1V} \neq \varphi_{2V}\). So \(\Gamma_{C_1, C_2}\) is injective and \(\Gamma\) is faithful.

\[\square\]

**Proposition 2** The functor \(\Gamma\) is an equivalence of categories \(L\text{-ChuCors}\) and \(L\text{-CLOS}\).

**Proof** Fullness and faithfulness of \(\Gamma\) is given by previous lemmas. Essential surjectivity on objects is ensured by the fact that given any object \(\langle\langle V, \approx\rangle, \preceq\rangle\) of \(L\text{-CLOS}\) there exists an \(L\)-context \(\langle V, V, \preceq\rangle\), such that \(\Gamma_{\langle V, V, \preceq\rangle} \sim \langle\langle V, \approx\rangle, \preceq\rangle\). Hence, we can state that \(\Gamma\) is the functor of equivalence between \(L\text{-ChuCors}\) and \(L\text{-CLOS}\).

\[\square\]

5 Conclusions and future work

After introducing the basic definitions concerning the \(L\)-fuzzy extension of formal concept analysis, as well as those concerning \(L\)-Chu correspondences, the categories associated to \(L\)-formal contexts and \(L\text{-CLOS}\) are defined and, finally, we provide a constructive proof of the equivalence between the categories of \(L\)-formal contexts with \(L\)-Chu correspondences as morphisms and that of completely lattice \(L\)-ordered sets and their corresponding morphisms. As a result, we obtain that the generalized approaches based on \(L\)-Chu correspondences and those on completely \(L\)-lattice ordered sets are mutually interchangeable.

Roughly similar results, in essence, have already been obtained, for instance, in [24]. In that paper, a new notion of morphism on formal contexts resulted in a category equivalent to both the category of complete algebraic lattices and Scott continuous functions, and a category of information systems and approximable mappings.

Other researchers have studied as well the relationships between Chu constructions and \(L\)-fuzzy FCA. For instance, in [14] FCA is linked to both order-theoretic developments in the theory of Galois connections and to Chu spaces; as a result, not surprisingly from our previous works, they obtain further relationships between formal contexts and topological systems within the category of Chu systems. Recently, Solovyov, in [41], extends the results of [14] to clarify the relationships between Chu spaces, many-valued formal contexts of FCA, lattice-valued interchange systems and Galois connections.

Potential applications are primary motivations for future work, for instance, to consider possible classes of formal \(L\)-contexts induced from existing datamining notions, and study its associated categories.
References