New links between mathematical morphology and fuzzy property-oriented concept lattices

J.C. Díaz, N. Madrid, J. Medina, and M. Ojeda-Aciego

***Abstract*—The theory of fuzzy property-oriented concept lattices is a formal tool for modeling and processing incomplete knowledge in information systems. This paper relates this research topic to that of mathematical morphology, a theory whose scope is to process and analyze images and signals. Consequently, the theory developed in the concept lattice framework can be used in these particular settings.**

I. Introduction

Fuzzy formal concept analysis and fuzzy rough set theory are two formal tools for modeling and processing incomplete information in information systems and, therefore, they are also used to extract information from these systems. The key notion that links these two approaches is that of the fuzzy property-oriented concept lattice framework [13], which arises as a fuzzy generalization of rough set theory and in which a set of objects and a set of attributes are assumed, following the view point of formal concept analysis.

On the other hand, mathematical morphology is a theory concerned with the processing and analysis of images or signals using filters and other operators that modify them, these morphological filters are obtained by means of two basic operators, the *dilation* and the *erosion*.

The fundamentals of this theory (initiated by G. Matheron [15] and J. Serra [19], [20]), are in set theory, integral geometry and lattice algebra. Actually, this methodology is used in general contexts related to activities such as information extraction in digital images, noise elimination or pattern recognition.

Recently, a first relationship between L-fuzzy concept lattices and fuzzy mathematical morphology was introduced in [1]. In the present work, the scope of this relationship is extended to the case of fuzzy property-oriented concept lattices.

Originally, the theory of formal concept analysis was developed on the basis of the properties of (antitone) Galois connections; later, further generalizations were introduced in terms of the closely related notion of isotone Galois connection. In fact, the latter notion had already been introduced in the framework of category theory under the name of adjunction (or pair of adjoint functors), and a number of its instances can be found in very disparate research areas. One of those areas is that of mathematical morphology, where the operators of erosion and dilation form an adjoint pair.

The crux of the links between both theories obtained in this paper is related to the fact that adjunctions (or, synonymously, isotone Galois connection) underlie both mathematical morphology and property-oriented multi-adjoint concept lattices.

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II. Preliminaries

This section recalls the fuzzy property-oriented concept lattices introduced in a more general environment than in [13], which use a generalization of the isotone Galois connections presented in [10]. Later, several basic notions of mathematical morphology are presented.

A. Fuzzy property-oriented concept lattices

In this section we recall a fuzzy generalization of the property-oriented concept lattices introduced in [16] and completed in [17]. The basic building blocks of this extension are the so-called adjoint triples, which consist of three operations: a non-commutativity conjunctor and two residuated implications [11], that satisfy the well-known adjoint property which allows to reproduce the *modus ponens* inference rule in a general framework.

Definition 1 ([18]): Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ be mappings, then $(\&, \swarrow, \nwarrow)$ is an *adjoint triple* with respect to P_1, P_2, P_3 if:

1) $\&$ is order-preserving in both arguments.
2) $\swarrow$ and $\nwarrow$ are order-preserving on the first argument[1] and order-reversing on the second argument.
3) $x \leq_1 z \swarrow y$ iff $x \,\&\, y \leq_3 z$ iff $y \leq_2 z \nwarrow x$, where $x \in P_1$, $y \in P_2$ and $z \in P_3$.

Gödel, product and Łukasiewicz t-norms, together with their residuated implications, can be seen as examples of adjoint triples.

Example 1: Since both Gödel, product and Łukasiewicz t-norms are commutative, the residuated implications satisfy that $\swarrow^G = \nwarrow_G$, $\swarrow^P = \nwarrow_P$ and $\swarrow^L = \nwarrow_L$. Therefore, the Gödel, product and Łukasiewicz adjoint triples are defined

[1] Note that the antecedent will be evaluated on the right side, while the consequent will be evaluated on the left side, as in logic programming framework.

on $[0,1]$ as usual, namely:

$$\&_P(x,y) = x \cdot y \quad ; \quad z \nwarrow_P x = \begin{cases} 1 & \text{if } x \le z \\ z/x & \text{otherwise} \end{cases}$$

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$$\begin{aligned} \&_L(x,y) &= \max\{0, x+y-1\} \\ z \nwarrow_L x &= \min\{1, 1-x+z\} \qquad \square \end{aligned}$$

More general examples of adjoint triples can be given. In the next example we show that the discretisations of the previous examples already lead to adjoint triples.

Example 2: Let $[0,1]_m$ be a regular partition of $[0,1]$ in m pieces, for example $[0,1]_2 = \{0, 0.5, 1\}$ divides the unit interval in two pieces.

Consider the discretization of the Gödel t-norm represented by the operator $\&_G^*\colon [0,1]_{20} \times [0,1]_8 \to [0,1]_{100}$ defined, for each $x \in [0,1]_{20}$ and $y \in [0,1]_8$, as:

$$x \,\&_G^*\, y = \frac{\lceil 100 \cdot \min\{x,y\} \rceil}{100}$$

where $\lceil \_ \rceil$ is the ceiling function.

For this operator, the corresponding residuated implication operators $\swarrow_G^*\colon [0,1]_{100} \times [0,1]_8 \to [0,1]_{20}$ and $\nwarrow_G^*\colon [0,1]_{100} \times [0,1]_{20} \to [0,1]_8$ are defined as:

$$\begin{aligned} b \swarrow_G^* a &= \frac{\lfloor 20 \cdot (b \swarrow^G a) \rfloor}{20} \\ b \nwarrow_G^* c &= \frac{\lfloor 8 \cdot (b \nwarrow_G a) \rfloor}{8} \end{aligned}$$

where $\lfloor \_ \rfloor$ is the floor function.

The tuple $(\&_G^*, \swarrow_G^*, \nwarrow_G^*)$ is an adjoint triple; it is remarkable that the operator $\&_G^*$ is neither commutative nor associative, as this is not required in the definition of adjoint triple. $\square$

The basic structure, which fixes the triplet of lattices and the adjoint triple, is the fuzzy property-oriented frame.

Definition 2: Given two complete lattices $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$, a poset $(P, \le)$ and one adjoint triple with respect to P, L_2, L_1, $(\&, \nwarrow)$, a *fuzzy property-oriented frame* is the tuple

$$(L_1, L_2, P, \preceq_1, \preceq_2, \le, \&, \nwarrow)$$

The notion of fuzzy property-oriented context is defined analogously to the one given in [18], and is given below.

Definition 3: Let $(L_1, L_2, P, \&, \nwarrow)$ be a fuzzy property-oriented frame. A *context* is a tuple (A, B, R) such that A and B are non-empty sets (usually interpreted as attributes and objects, respectively), R is a P-fuzzy relation $R\colon A \times B \to P$.

From now on, we will consider a fixed fuzzy property-oriented frame $(L_1, L_2, P, \&, \nwarrow)$ and a fixed context (A, B, R).

The mappings ${}^{\uparrow_\Pi}\colon L_2^B \to L_1^A$ and ${}^{\downarrow^N}\colon L_1^A \to L_2^B$ are defined as

$$\begin{aligned} g^{\uparrow_\Pi}(a) &= \bigvee\{R(a,b) \,\&\, g(b) \mid b \in B\} \\ f^{\downarrow^N}(b) &= \bigwedge\{f(a) \nwarrow R(a,b) \mid a \in A\} \end{aligned}$$

These definitions generalize the classical possibility and necessity operators [9]. Moreover, $({}^{\uparrow_\Pi}, {}^{\downarrow^N})$ is an isotone Galois connection (also known as adjunction) and, therefore, ${}^{\uparrow_\Pi\downarrow^N}\colon L_2^B \to L_2^B$ is a closure operator and ${}^{\downarrow^N\uparrow_\Pi}\colon L_1^A \to L_1^A$ is an interior operator.

A concept, in this environment, is a pair of mappings $\langle g, f \rangle$, with $g \in L^B, f \in L^A$, such that $g^{\uparrow_\Pi} = f$ and $f^{\downarrow^N} = g$, which will be called *fuzzy property-oriented concept*. In that case, g is called the *extent* and f, the *intent* of the concept. The set of all these concepts will be denoted as $\mathcal{F}_{\Pi N}$.

Definition 4: The associated *fuzzy property-oriented concept lattice* to the fixed frame and context (or, the *concept lattice of* (A, B, R) *based on rough set theory*) is defined as the set

$$\mathcal{F}_{\Pi N} = \{\langle g, f \rangle \in L_2^B \times L_1^A \mid g^{\uparrow_\Pi} = f \text{ and } f^{\downarrow^N} = g\}$$

in which the ordering is defined by $\langle g_1, f_1 \rangle \preceq \langle g_2, f_2 \rangle$ iff $g_1 \preceq_2 g_2$ (or equivalently $f_1 \preceq_1 f_2$).

The pair $(\mathcal{F}_{\Pi N}, \preceq)$ is a complete lattice [16], which generalizes the concept lattice introduced in [6] to a fuzzy environmernt.

B. Mathematical Morphology

The fundamentals of mathematical morphology ground on two basic operators: erosions and dilations. Those operators were introduced originally on Euclidean Spaces by means of translations and joins of subsets [15], [19]. However, in subsequent approaches [12], [20], such definitions were extended to apply to complete lattices in order to cover broader applications. In this paper, we use directly this later algebraical definition.

Definition 5: Let $(L_1, \le_1)$ and $(L_2, \le)$ be two complete lattices. A mapping $\varepsilon\colon L_1 \to L_2$ is called an *erosion* if for all $X \subseteq L_1$ we have:

$$\varepsilon(\bigwedge X) = \bigwedge_{x \in X} \varepsilon(x)$$

A mapping $\delta\colon L_2 \to L_1$ is called a *dilation* if for all $Y \subseteq L_2$ we have:

$$\delta(\bigvee Y) = \bigvee_{y \in Y} \delta(y)$$

So, roughly speaking, every erosion commutes with infimum and every dilation with supremum. Note that the definition above takes into account the case where X and Y are empty. That means that erosions assign the greatest element of L_1 to the greatest element of L_2 and dilations assign the least element of L_2 to the least element of L_1. Let us see some examples.

Example 3: The best known family of erosions and dilations comes from the original definition in the framework of

Euclidean Spaces. Specifically, the lattice considered here is the powerset of the real plane $\mathbb{R}^2$ (denoted by $\wp\mathbb{R}^2$). Such definitions are based on translations, so let us recall that for each $P \in \mathbb{R}^2$, we can consider the translation τ_P defined as the only translation which sends the origin $O \in \mathbb{R}^2$ to P. Thus, given a subset $S \subseteq \mathbb{R}^2$, called *structuring element*, the erosion ε_S and dilation δ_S of a subset $X \subseteq \mathbb{R}^2$ are defined by

$$\varepsilon_S(X) = \bigcap_{s\in S} \tau_{-s}(X)$$

$$\delta_S(X) = \bigcup_{s\in S} \tau_s(X)$$

It is well-known that ε_S and δ_S defined as above are really an erosion and a dilation, respectively, according to Definition 5. □

Example 4: The family of erosions and dilations given in the example above are invariant under translation. Notice, however, that it is easy to define erosions and dilations without such a property in $\wp\mathbb{R}^2$. Consider a mapping $W: \mathbb{R}^2 \to \wp\mathbb{R}^2$ associating to each point a subset of $\mathbb{R}^2$. Then, for each subset X of $\mathbb{R}^2$, we can introduce the operator δ_W defined by

$$\delta_W(X) = \bigcup_{x\in X} W(x)$$

which is a dilation.

We can obtain an erosion from δ_W by complementation, that is, by defining $\varepsilon_W(X) = (\delta_W(X^c))^c$. □

Perhaps the most important relation between erosions and dilations is given in terms of adjoint pairs, so let us begin by recalling the notion of adjoint pair.

Definition 6: A pair (ε, δ) of mappings $\varepsilon: L_1 \to L_2$ and $\delta: L_2 \to L_1$ between complete lattices $(L_1, \leq_1)$ and $(L_2, \leq)$ is an adjoint pair if and only if for every $X \in L_1$ and $Y \in L_2$ we have

$$Y \leq \varepsilon(X) \quad \text{if and only if} \quad \delta(Y) \leq X$$

The naming ε and δ chosen in the previous definitions is not casual, since the operators introduced in Definitions 5 and 6 are adjoint each other. The following two results (Theorems 1 and 2) were proven in [12], somehow rediscovering facts well-known in category theory. Sadly, such a situation is repeatedly found in different areas.

Firstly, we have the following result:

Theorem 1: If (ε, δ) an adjoint pair. Then, ε is an erosion and δ is a dilation.

On the other hand, the converse can be written in the following sense:

Theorem 2: Let $\varepsilon: L_1 \to L_2$ be an erosion. Then there exists exactly one dilation $\delta_\varepsilon: L_2 \to L_1$ such that $(\varepsilon, \delta_\varepsilon)$ forms an adjoint pair. Specifically, such a dilation can be determined by the expression:

$$\delta_\varepsilon(Y) = \bigwedge\{Z \in L_2 \mid Y \leq \varepsilon(Z)\}$$

for every $Y \in L_2$.

Similarly, for every dilation $\delta: L_2 \to L_1$ there exists exactly one erosion $\varepsilon_\delta: L_1 \to L_2$ such that $(\varepsilon_\delta, \delta)$ forms an adjoint pair. Moreover, such an erosion is determined by the expression:

$$\varepsilon_\delta(X) = \bigvee\{Z \in L_1 \mid \delta(Z) \leq X\}$$

for every $X \in L_1$.

Example 5: The operators ε_S and δ_S defined in Example 3 form an adjoint pair independently of the structuring element $S \subseteq \mathbb{R}^2$. □

It is worth mentioning that Mathematical Morphology is not just about considering erosions and dilations; many other notions and operations are used as well, for instance *openings, closings* and *Hit-Miss transformations*, among others, are also object of study in this theory. But for the sake of simplicity, we restrict the introduction of preliminary notions to a minimum and will not go further in this section.

III. A FIRST BRIDGE BETWEEN BOTH THEORIES

In some sense, a first link between both theories can be straightforwardly obtained from Theorem 2, using the fact that the necessity and possibility operators forms an adjunction (or isotone Galois connection). Therefore, we can formulate the following proposition:

Proposition 1: Every necessity (respectively, possibility) operator of a property-oriented concept lattice is an erosion (resp., dilation).

The converse is also true in some sense. That is, any erosion operator ε can be identified with the necessity operator associated to some fuzzy property-oriented concept lattice. However the proof is not as straightforward as in the previous result.

Theorem 3: Consider two complete lattices, $\overline{L}_1$ and $\overline{L}_2$, and an erosion operator $\varepsilon: \overline{L}_1 \to \overline{L}_2$, then there exists a frame $(L_1, L_2, P, \&, \nwarrow)$, a context (A, B, R) and two isomorphisms $\phi_1: \overline{L}_1 \to L_1^A$, $\phi_2: \overline{L}_2 \to L_2^B$, such that $\varepsilon = \phi_2^{-1} \circ {\downarrow^N} \circ \phi_1$, where ${\downarrow^N}$ is the necessity operator associated with the frame and context.

Proof: To begin with, we will build the frame. The choice of lattices L_1 and L_2 is easy, we consider $L_1 = \overline{L}_1$ and $L_2 = \overline{L}_2$; the poset P is the singleton $\{e\}$, where e is an arbitrary element and the implication $\nwarrow: L_1 \times P \to L_2$ is defined, for all $z \in L_1$, as $z \nwarrow e = \varepsilon(z)$.

The adjoint conjunctor $\&: P \times L_2 \to L_1$ is defined from the implication as:

$$\begin{aligned} e \,\&\, y &= \bigwedge\{z \in L_1 \mid y \preceq z \nwarrow x\} \\ &= \bigwedge\{z \in L_1 \mid y \preceq \varepsilon(z)\} = \delta(y) \end{aligned}$$

for all $y \in L_2$, where $\delta: L_2 \to L_1$ is the dilation satisfying that (ε, δ) is an isotone Galois connection. Consequently, the pair $(\&, \nwarrow)$ satisfies the adjoint property and, therefore, the considered property-oriented frame is $(\overline{L}_1, \overline{L}_2, P, \&, \nwarrow)$.

Now, the context (A, B, R) is given by $A = B = \{a\}$ and $R: A \times B \to P$ is defined by $R(a, a) = e$.

Since A is a singleton, we can give a trivial isomorphism $\phi_1: \overline{L}_1 \to L_1^A$, defined for every $z \in L_1$, as $\phi_1(z) = f_z$, where $f_z \in L_1^A$ is the constant mapping $f_z(a) = z$. Analogously, an isomorphism $\phi_2 \colon \overline{L}_2 \to L_2^B$ can be introduced. Note that, given $g \in L_2^B$, $\phi_2^{-1}(g) = g(a)$.

Thus, given $z \in L_1$, we have the following chain of equalities

$$\begin{aligned} \phi_2^{-1} \circ {\downarrow^N} \circ \phi_1(z) &= \phi_2^{-1}(f_z{\downarrow^N}) \\ &= f_z{\downarrow^N}(a) \\ &= \bigwedge\{f_z(a) \nwarrow R(a,a) \mid a \in A\} \\ &= f_z(a) \nwarrow R(a,a) \\ &= \varepsilon(f_z(a)) \\ &= \varepsilon(z) \end{aligned}$$

■

Note that the theorem above can be rewritten in terms of dilations as follows:

Corollary 1: Given two complete lattices, $\overline{L}_1$ and $\overline{L}_2$, and a dilation operator $\varepsilon \colon \overline{L}_2 \to \overline{L}_2$, there exists a frame $(L_1, L_2, P, \&, \nwarrow)$, a context (A, B, R) and two isomorphisms $\phi_1 \colon \overline{L}_1 \to L_1^A$, $\phi_2 \colon \overline{L}_2 \to L_2^B$, such that $\delta = \phi_1^{-1} \circ {\uparrow_\Pi} \circ \phi_2$, where ${\uparrow_\Pi}$ is the necessity operator associated with the frame and context.

The theorems introduced so far establish a strong link between the theory of fuzzy property-oriented concept lattices and mathematical morphology. However, although the existence of the link is interesting theoretically, it could be useless in practice. The reason is because the context built in the proof of Theorem 3 simply consists of two singletons and a trivial relation.

In the rest of the section, we focus on a slightly different problem which can be stated as follows: given an isotone Galois connection (or adjunction) (ε, δ) with $\varepsilon \colon {L_1}^A \to {L_2}^B$, we study the existence of a frame $(L_1, L_2, P, \&, \nwarrow)$ and context (A, B, R) satisfying the equality $f^{\downarrow^N} = \varepsilon(f)$ and $g^{\uparrow_\Pi} = \delta(g)$.

It is worth to point out two interesting and important differences of this statement with the one given in Theorem 3. On the one hand, both the domain and codomain of the erosion (and, obviously, the dilation) are fuzzy powersets, which are directly related to the frame and context of the concept lattice, thus they are already known elements. In other words, to prove the statement, we simply need to define the poset P together with the conjunction $\&$ and implication $\nwarrow$, and the relation $R \colon A \times B \to P$. And on the other hand, the relationship between the necessity operator and the erosion does not involve any isomorphism, instead they are exactly the same.

The answer to the problem stated above is positive, and the detailed construction of the frame and context are given in the proof of the following theorem.

Theorem 4: Let $\varepsilon \colon {L_1}^A \to {L_2}^B$ be an erosion, then there exists a frame $(L_1, L_2, P, \&, \nwarrow)$ and context (A, B, R) satisfying the equality $f^{\downarrow^N} = \varepsilon(f)$.

Proof: To begin with, concerning the frame, we choose the poset P to be the Cartesian product with the discrete ordering, that is the ordering defined by the equality relation, i.e. $(P, \leq_3) = (A \times B, =)$. As a result of this choice, one can define the relation $R \colon A \times B \to A \times B$ as the identity mapping.

Now, in order to define the implication $\nwarrow \colon L_1 \times (A \times B) \to L_2$, it will be convenient to consider the family of auxiliary mappings introduced below:

Given an attribute $a_0 \in A$ and a value $x \in L_1$, the mapping $\phi_{a_0,x} \colon A \to L_1$ is defined as

$$\phi_{a_0,x}(a) = \begin{cases} x & \text{if } a = a_0 \\ \top & \text{otherwise} \end{cases}$$

Now, given $x \in L_1$ and $(a, b) \in A \times B$, we define

$$x \nwarrow (a, b) = \varepsilon(\phi_{a,x})(b)$$

which is increasing in the first argument (this is easy, since the mappings $\phi_{a_i,x}$ are increasing wrt x, and also ε is increasing) and, vacuously, decreasing in the second component (since we are considering the discrete ordering in $A \times B$).

At this point, aiming at the expression given in the statement, it is not difficult to see that its "expected" associated conjunctor $\& \colon (A \times B) \times L_2 \to L_1$ by

$$\begin{aligned} (a, b) \mathbin{\&} y &= \bigwedge\{z \mid y \leq_2 z \nwarrow (a, b)\} \\ &= \bigwedge\{z \mid y \leq_2 \varepsilon(\phi_{a,z})(b)\} \end{aligned}$$

In order to check that, actually, $\&$ and $\nwarrow$ satisfy the corresponding adjoint property, one has to check that the infimum in the definition above is indeed a minimum. This follows from the fact that the meet of elements z_j satisfying $y \leq_2 \varepsilon(\phi_{a,z_j})(b)$ actually satisfies the inequality as well, since

$$\varepsilon(\phi_{a,\bigwedge_j z_j}) = \varepsilon(\bigwedge_j \phi_{a,z_j}) = \bigwedge_j \varepsilon(\phi_{a,z_j})$$

We have just to verify that the necessity operator associated to the frame and context just described coincides with ε. Firstly, we will check the equality for the ϕ-mappings:

$$\begin{aligned} \phi_{a_i,x}^{\downarrow^N}(b) &= \bigwedge_{a \in A} \phi_{a_i,x}(a) \nwarrow R(a, b) \\ &= \phi_{a_i,x}(a_i) \nwarrow R(a_i, b) \\ &= x \nwarrow R(a_i, b) \\ &= x \nwarrow (a_i, b) \\ &= \varepsilon(\phi_{a_i,x})(b) \end{aligned}$$

Finally, given any mapping $f \in {L_1}^A$, we will benefit from another use of the ϕ-mappings as a means to represent f as a meet indexed by the elements of A. Specifically, we can write $f = \bigwedge_{i \in I} \phi_{a_i, f(a_i)}$ where I is an index set such that

$A = \{a_i \mid i \in I\}$. Thus,

$$\begin{aligned} f^{\downarrow^N}(b) &= \Big(\bigwedge_{i\in I} \phi_{a_i, f(a_i)}\Big)^{\downarrow^N}(b) \\ &= \bigwedge_{i\in I} \phi_{a_i, f(a_i)}{}^{\downarrow^N}(b) \\ &= \bigwedge_{i\in I} \varepsilon(\phi_{a_i, f(a_i)})(b) \\ &= \varepsilon\Big(\bigwedge_{i\in I} \phi_{a_i, f(a_i)}\Big)(b) \\ &= \varepsilon(f)(b) \end{aligned}$$

■

IV. Conclusions and future work

Focusing on the fact that adjunctions (or isotone Galois connections) underlie the theory of both property-oriented multi-adjoint concept lattices and mathematical morphology, we have obtained some results linking erosion and dilation operators to the necessity operator of certain instance of property-oriented multi-adjoint concept lattice.

The existence of this first link allows for foreseeing future developments in which both frameworks can be merged, so that algorithms given for fuzzy concept lattices could be applied to mathematical morphology and vice versa.

In this paper, we have established a relationship between certain framework of fuzzy formal concept analysis and mathematical morphology. Concerning possible future work, we will further extend the link by considering extended frameworks in both topics. On the one hand, new theoretical results have been obtained concerning the use of Galois connections within multi-adjoint concept lattices [8]; on the other hand, we can consider as well fuzzy mathematical morphology, which has interesting applications, for instance, *fuzzy morphological image processing* has been given in [2]–[5], [7], [14] using L-fuzzy sets as images and structuring elements.

References

[1] C. Alcalde, A. Burusco, J.C. Díaz, R. Fuentes-González, and J. Medina-Moreno. Fuzzy property-oriented concept lattices in morphological image and signal processing. *Lecture Notes in Computer Science*, 7903:246–253, 2013.

[2] I. Bloch. Duality vs. adjunction for fuzzy mathematical morphology and general form of fuzzy erosions and dilations. *Fuzzy Sets and Systems*, 160(0):1858–1867, 2009.

[3] I. Bloch and H. Maître. Fuzzy mathematical morphologies: a comparative study. *Télécom Paris 94D001*, 1994.

[4] P. Burillo, N. Frago, and R. Fuentes-González. Generation of Fuzzy Mathematical Morphologies. *Mathware & Soft Computing*, 8(1):31–46, 2001.

[5] P. Burillo, R. Fuentes-González, and N. Frago. Inclusion grade and fuzzy implication operators. *Fuzzy Sets and Systems*, 114(3):417–429, 2000.

[6] Y. Chen and Y. Yao. A multiview approach for intelligent data analysis based on data operators. *Information Sciences*, 178(1):1–20, 2008.

[7] B. De Baets. Fuzzy morphology: a logical approach. In B. Ayyub and M. Gupta, editors, *Uncertainty Analysis, Engineering and Science: Fuzzy Logic, Statistics and neural network Approach*, pages 53–67. Kluwer Academic Publishers, 1997.

[8] J.C. Díaz, J. Medina, and M. Ojeda-Aciego. On basic conditions to generate multi-adjoint concept lattices via Galois connections. *Intl Journal of General Systems*, 2014. Accepted.

[9] G. Gediga and I. Düntsch. Modal-style operators in qualitative data analysis. In *Proc. IEEE Int. Conf. on Data Mining*, pages 155–162, 2002.

[10] G. Georgescu and A. Popescu. Non-dual fuzzy connections. *Arch. Math. Log.*, 43(8):1009–1039, 2004.

[11] P. Hájek. *Metamathematics of Fuzzy Logic*. Trends in Logic. Kluwer Academic, 1998.

[12] H.J.A.M. Heimans and C. Ronse. The Algebraic Basis of Mathematical Morphology. I. Dilations and Erosions *Computer Vision, Graphics, and Image Processing* 50(3):245–295, 1990.

[13] H. Lai and D. Zhang. Concept lattices of fuzzy contexts: Formal concept analysis vs. rough set theory. *International Journal of Approximate Reasoning*, 50(5):695–707, 2009.

[14] P. Maragos. Lattice image precessing: A unification of morphological and fuzzy algebric systems. *Journal of Mathematical Imaging and Vision*, 22:333–353, 2005.

[15] G. Matheron. *Random Sets and Integral Geometry*. Wiley, 1975.

[16] J. Medina. Towards multi-adjoint property-oriented concept lattices. *Lecture Notes in Artificial Intelligence*, 6401:159–166, 2010.

[17] J. Medina. Multi-adjoint property-oriented and object-oriented concept lattices. *Information Sciences*, 190:95–106, 2012.

[18] J. Medina, M. Ojeda-Aciego, and J. Ruiz-Calviño. Formal concept analysis via multi-adjoint concept lattices. *Fuzzy Sets and Systems*, 160(2):130–144, 2009.

[19] J. Serra. *Image Analysis and Mathematical Morphology: Vol. 1*. Academic Press, 1982.

[20] J. Serra. *Image Analysis and Mathematical Morphology: Vol. 2. Theoretical Advances*. Academic Press, 1988.