A fixed-point theorem for multi-valued functions with an application to multilattice-based logic programming

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Abstract. This paper presents a computability theorem for fixed points of multi-valued functions defined on multilattices, which is later used in order to obtain conditions which ensure that the immediate consequence operator computes minimal models of multilattice-based logic programs in at most ω iterations.

1 Introduction

Following the trend of generalising the structure of the underlying set of truth-values for fuzzy logic programming, multilattice-based logic programs were introduced in [7] as an extended framework for fuzzy logic programming, in which the underlying set of truth-values for the propositional variables is considered to have a more relaxed structure than that of a complete lattice.

The first definition of multilattices, to the best of our knowledge, was introduced in [1], although, much later, other authors proposed slightly different approaches [4,6]. The crucial point in which a complete multilattice differs from a complete lattice is that a given subset does not necessarily have a least upper bound (resp. greatest lower bound) but some minimal (resp. maximal) ones.

As far as we know, the first paper which used multilattices in the context of fuzzy logic programming was [7], which was later extended in [8]. In these papers, the meaning of programs was defined by means of a fixed point semantics; and the non-existence of suprema in general but, instead, a set of minimal upper bounds, suggested the possibility of developing a non-deterministic fixed point theory in the form of a multi-valued immediate consequences operator.

Essentially, the results presented in those papers were the existence of minimal models below any model of a program, and that any minimal model can be attained by a suitable version of the iteration of the immediate consequence operator; but some other problems remained open, such as the constructive nature of minimal models or the reachability of minimal models after at most countably many iterations.

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The aim of this paper is precisely to present conditions which ensure that minimal models for multilattice-based logic programs can be reached by a "bounded" iteration of the immediate consequences operator, in the sense that fixed points are attained after no more than ω iterations. Obviously, the main theoretical problem can be stated in the general framework of multi-valued functions on a multilattice. Some existence results in this line can be found in [2,3,5,9,10], but they worked with complete lattices instead of multilattices.

The structure of the paper is as follows: in Section 2, some preliminary definitions and results are presented; later, in Section 3, we introduce the main contribution of the paper, namely, reachability results for minimal fixed points of multi-valued functions on a multilattice; then, in Section 4, these results are instantiated to the particular case of the immediate consequences operator of multilattice-based logic programs; the paper finishes with some conclusions and prospects for future work.

2 Preliminaries

In order to make this paper self-contained, we provide in this section the basic notions of the theory of multilattices, together with a result which will be used later. For further explanations, the reader can see [7,8].

Definition 1. *A* complete multilattice *is a partially ordered set, $\langle M, \preceq \rangle$, such that for every subset $X \subseteq M$, the set of upper (resp. lower) bounds of X has minimal (resp. maximal) elements, which are called* multi-suprema *(resp.* multi-infima*).*

The sets of multi-suprema and multi-infima of a set X are denoted by $\text{multisup}(X)$ and $\text{multinf}(X)$. It is straightforward to note that these sets consist of pairwise incomparable elements (also called *antichains*).

An upper bound of a set X needs not be greater than any minimal upper bound (multi-supremum); such a condition (and its dual, concerning lower bounds and multi-infima) has to be explicitly required. This condition is called *coherence*, and is formally introduced in the following definition, where we use the Egli-Milner ordering , i.e., $X \sqsubseteq_{EM} Y$ if and only if for every $y \in Y$ there exists $x \in X$ such that $x \preceq y$ and for every $x \in X$ there exists $y \in Y$ such that $x \preceq y$.

Definition 2. *A complete multilattice M is said to be* coherent *if the following pair of inequations hold for all $X \subseteq M$:*

$$LB(X) \sqsubseteq_{EM} \text{multinf}(X)$$
$$\text{multisup}(X) \sqsubseteq_{EM} UB(X)$$

where $LB(X)$ and $UB(X)$ denote, respectively, the sets of lower bounds and upper bounds of the set X.

Coherence together with the non-existence of infinite antichains (so that the sets multisup(X) and multinf(X) are always finite) have been shown to be useful conditions when working with multilattices. Under these hypotheses, the following important result was obtained in [7]:

Lemma 1. *Let M be a coherent complete multilattice without infinite antichains, then any chain*[1] *in M has a supremum and an infimum.*

3 Reaching Fixed Points for Multi-valued Functions on Multilattices

In order to proceed to the study of existence and reachability of minimal fixed points for multi-valued functions, we need some preliminary definitions.

Definition 3. *Given a poset P, by a* multi-valued function *we mean a function $f\colon P \longrightarrow 2^P$ (we do not require that $f(x) \neq \varnothing$ for every $x \in P$).*

We say that $x \in P$ is a fixed point *of f if and only if $x \in f(x)$.*

The adaptation of the definition of isotonicity and inflation for multi-valued functions is closely related to the ordering that we consider on the set 2^M of subsets of M. We will consider the Smyth ordering among sets, and we will write $X \sqsubseteq_S Y$ if and only if for every $y \in Y$ there exists $x \in X$ such that $x \preceq y$.

Definition 4. *Let $f\colon P \longrightarrow 2^P$ be a multi-valued function on a poset P:*

- *We say that f is* isotone *if and only if for all $x, y \in P$ we have that $x \preceq y$ implies $f(x) \sqsubseteq_S f(y)$.*
- *We say that f is* inflationary *if and only if $\{x\} \sqsubseteq_S f(x)$ for every $x \in P$.*

As our intended application is focused on multilattice-based logic programs, we can assume the existence of minimal fixed points for a given multi-valued function on a multilattice (since in [7] the existence of minimal fixed points was proved for the $T_{\mathbb{P}}$ operator). Regarding reachability of a fixed point, it is worth to rely on the so-called *orbits* [5]:

Definition 5. *Let $f\colon M \longrightarrow 2^M$ be a multi-valued function an* orbit of f *is a transfinite sequence $(x_i)_{i\in I}$ of elements $x_i \in M$ where the cardinality of M is less than the cardinality of I ($|M| < |I|$) and:*

$$\begin{aligned} x_0 &= \bot \\ x_{i+1} &\in f(x_i) \\ x_\alpha &\in \text{multisup}\{x_i \mid i < \alpha\} \text{ , for limit ordinals } \alpha \end{aligned}$$

Note the following straightforward consequences of the definition:

[1] A chain X is a totally ordered subset. Sometimes, for convenience, a chain will be denoted as an indexed set $\{x_i\}_{i\in I}$.

1. In an orbit, we have $f(x_i) \neq \varnothing$ for every $i \in I$.
2. As $f(x_i)$ is a nonempty set, there might be many possible choices for x_{i+1}, so we might have many possible orbits.
3. If $(x_i)_{i\in I}$ is an orbit of f and there exists $k \in I$ such that $x_k = x_{k+1}$, then x_k is a fixed point of f.
   Providing sufficient conditions for the existence of such orbits, we ensure the existence of fixed points. Note that the condition $f(\top) \neq \varnothing$ directly implies the existence of a fixed point, namely $\top$.
4. Any increasing orbit eventually reaches a fixed point (this follows from the inequality $|M| < |I|$).
   This holds because every transfinite increasing sequence is eventually stationary, and an ordinal α such that $x_\alpha = x_{\alpha+1} \in f(x_\alpha)$ is a fixed point.

Under the assumption of f being non-empty and inflationary, the existence of increasing orbits can be guaranteed; the proof is roughly sketched below:

The orbit can be constructed for any successor ordinal α by using the inequality $\{x_\alpha\} \sqsubseteq_S f(x_\alpha)$, which follows by inflation, since any element $x_{\alpha+1} \in f(x_\alpha)$ satisfies $x_\alpha \preceq x_{\alpha+1}$. The definition for limit ordinals, directly implies that it is greater than any of its predecessors.

As a side result, note that when reaching a limit ordinal, under the assumption of f being inflationary, the initial segment is actually a chain; therefore, by Lemma 1 it has only one multi-supremum (the supremum of the chain); this fact will be used later in Propositions 1 and 2.

Regarding minimal fixed points, the following result shows conditions under which any minimal fixed point is attained by means of an orbit:

Proposition 1. *Let $f\colon M \longrightarrow 2^M$ be inflationary and isotone, then for any minimal fixed point there is an orbit converging to it.*

Proof. Let x be a minimal fixed point of f and let us prove that there is an increasing orbit $(x_i)_{i\in I}$ satisfying $x_i \preceq x$. We will build this orbit by transfinite induction:

Trivially $x_0 = \bot \preceq x$.

If $x_i \preceq x$, by isotonicity $f(x_i) \sqsubseteq_S f(x)$. Then for $x \in f(x)$ we can choose $x_{i+1} \in f(x_i)$ such that $x_{i+1} \preceq x$ and obviously $x_i \preceq x_{i+1}$ by inflation.

For a limit ordinal α, as stated above, $x_\alpha = \sup_{i<\alpha} x_i$; now, by induction we have that $x_i \preceq x$ for every $i < \alpha$, hence $x_\alpha \preceq x$.

The transfinite chain $(x_i)_{i\in I}$ constructed this way is increasing, therefore there is an ordinal α such that $x_\alpha = x_{\alpha+1} \in f(x_\alpha)$, so x_α is a fixed point and $x_\alpha \preceq x$ but by minimality of the fixed point x, we have that $x = x_\alpha$. □

The usual way to approach the problem of reachability is to consider some kind of 'continuity' in our multi-valued functions, understanding continuity in the sense of preservation of suprema and infima. But it is obvious that we have to state formally what this preservation is meant, since in complete multilattices we only have for granted the existence of *sets of multi-infima* and *sets of multi-suprema*. This is just another reason to rely on coherent complete multilattices M

without infinite antichains so that, at least, we have the existence of suprema and infima of chains.

Definition 6. *A multi-valued function* $f\colon M \longrightarrow 2^M$ *is said to be* sup-preserving *if and only if for every chain* $X = (x_i)_{i\in I}$ *we have that:*

$$f(\sup\{x_i \mid i \in I\}) = \{y \mid \text{ there are } y_i \in f(x_i) \text{ s.t. } y \in \operatorname{multisup}\{y_i \mid i \in I\}\}$$

Note that, abusing a bit the notation, the definition above can be rephrased in much more usual terms as $f(\sup X) = \operatorname{multisup}(f(X))$ but we will not use it, since the intended interpretation of $\operatorname{multisup}(f(X))$ is by no means standard.

Reachability of minimal fixed points is granted by assuming the extra condition that our function f is sup-preserving, as shown in the following proposition.

Proposition 2. *If a multi-valued function f is inflationary, isotone and sup-preserving, then at most countably many steps are necessary to reach a minimal fixed point (provided that some exists).*

Proof. Let x be a minimal fixed point and consider the approximating increasing orbit $(x_i)_{i\in I}$ given by Proposition 1. We will show that x_ω is a fixed point of f and, therefore, x_ω equals x.

As f is sup-preserving we have that $f(x_\omega)$ is the set

$$\{y \mid \text{ there are } y_i \in f(x_i) \text{ s.t. } y = \operatorname{multisup}\{y_i \mid i < \omega\}\}$$

In order to prove that x_ω is a fixed point, on the one hand, recall that we have, by definition, that $x_\omega = \sup\{x_i \mid i < \omega\}$. On the other hand, we will show that this construction can be also seen as a multi-supremum of a suitable sequence of elements $y_i \in f(x_i)$.

To do this we only have to recall that, by construction of the orbit, we know that $x_{i+1} \in f(x_i)$, therefore for every $0 \le i < \omega$ we can consider $y_i = x_{i+1}$. Hence the element x_ω can be seen as an element of $f(x_\omega)$. Thus, x_ω is a fixed point of f and $x_\omega \preceq x$ and by minimality of x, we have that $x = x_\omega$. □

4 Application to fuzzy logic programs on a multilattice

In this section we apply the previous results to the particular case of the immediate consequences operator for extended logic programs on a multilattice, as defined in [7, 8]. To begin with, we will assume the existence of a multilattice (coherent and without infinite antichains) M as the underlying set of truth-values, that is, our formulas will have certain degree of truth in M. In order to build our formulas, we will consider a set of computable n-ary isotone operators $M^n \longrightarrow M$ which will be intended as our logical connectors. Finally, we will consider a set Π of propositional symbols as the basic blocks which will allow to build the set of formulas, by means of the connector functions.

Now, we can recall the definition of the fuzzy logic programs based on a multilattice:

Definition 7. *A* fuzzy logic program based on a multilattice M *is a set* $\mathbb{P}$ *of rules of the form* $A \leftarrow \mathcal{B}$ *such that:*

1. *A is a propositional symbol of Π, and*
2. *$\mathcal{B}$ is a formula built from propositional symbols and elements of M by using isotone operators.*

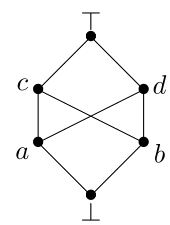
Now we give the definition of interpretation and model of a program:

Definition 8.

1. *An interpretation is a mapping $I \colon \Pi \longrightarrow M$.*
2. *We say that I satisfies a rule $A \leftarrow \mathcal{B}$ if and only if $\hat{I}(\mathcal{B}) \leq I(A)$, where $\hat{I}$ is the homomorphic extension of I to the set of all formulae.*
3. *An interpretation I is said to be a* model *of a program $\mathbb{P}$ iff all rules in $\mathbb{P}$ are satisfied by I.*

Example 1. Let us consider an example of a program on a multilattice. The program consists of the four rules below to the left, whereas the underlying multilattice is the six-element multilattice depicted below to the right:

$$E \leftarrow A$$
$$E \leftarrow B$$
$$A \leftarrow a$$
$$B \leftarrow b$$



It is easy to check that the program does not have a least model but two minimal ones, I_1 and I_2, given below:

$$\begin{array}{ll} I_1(E) = c & I_2(E) = d \\ I_1(A) = a & I_2(A) = a \\ I_1(B) = b & I_2(B) = b \end{array} \qquad \square$$

A fixed point semantics was given by means of the following consequences operator:

Definition 9. *Given a fuzzy logic program $\mathbb{P}$ based on a multilattice M, an interpretation I and a propositional symbol A; the immediate consequences operator is defined as follows:*

$$T_{\mathbb{P}}(I)(A) = \text{multisup}\left(\{I(A)\} \cup \{\hat{I}(\mathcal{B}) \mid A \leftarrow \mathcal{B} \in \mathbb{P}\}\right)$$

Note that, by the very definition, the immediate consequences operator is an inflationary multi-valued function defined on the set of interpretations of the program $\mathbb{P}$, which inherits the structure of multilattice. Moreover, models can be characterized in terms of fixed points of $T_{\mathbb{P}}$ as follows:

Proposition 3 (see [7]). *An interpretation I is a model of a program if and only if $I \in T_{\mathbb{P}}(I)$.*

Although not needed for the definition of either the syntax or the semantics of fuzzy logic programs, the requirement that M is a coherent multilattice without infinite antichains turns out to be essential for the existence of minimal fixed points, see [7]. Hence, a straightforward application of Proposition 2 allows us to obtain the following result.

Theorem 1. *If $T_{\mathbb{P}}$ is sup-preserving, then ω steps are sufficient to reach a minimal model.*

5 Conclusions

Continuing the study of computational properties of multilattices initiated in [7], we have presented a theoretical result regarding the attainability of minimal fixed points of multi-valued functions on a multilattice which, as an application, guarantees that minimal models of multilattice-based logic programs can be attained after at most countably many iterations of the immediate consequence operator. We recall that, in this paper, the existence of such fixed points has been assumed because of the intended application in mind (that is, the existence of minimal models for multilattice-based logic programs was proved in [7]).

As future work, this initial investigation on fixed points of multi-valued functions on a multilattice has to be completed with the study of sufficient conditions for the existence of (minimal) fixed points.

Another interesting line of research, which turns out to be fundamental for the practical applicability of the presented result, is the study of conditions which guarantee that the immediate consequences operator is sup-preserving.

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